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LETTER TO THE EDITOR

# Variation of the action in the classical time-dependent harmonic oscillator: an exact result

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**Abstract.** From the exact solution of certain time-dependent classical harmonic oscillators in one dimension we investigate the behaviour of the adiabatic invariant $J(t)$. For a subclass of such potentials $\Delta J \equiv J(+\infty) - J(-\infty) = 0$ whatever the regime. We show that this does not necessarily imply a breakdown of the commonly accepted asymptotic exponential law for $\Delta J$.

Although the classical harmonic oscillator (HO) with time-dependent frequency is a very old subject of study in applied mathematics and physics it continues to receive attention nowadays, often as a way of checking the validity of new approximation methods, but also to illustrate ideas or concepts. It is therefore not surprising that original remarks concerning the time-dependent HO appear regularly in the literature. The present note has been motivated by one of these recent observations [1–3].

To be precise, in the first part of this letter we solve exactly the equation of motion associated to the one-dimensional time-dependent Hamiltonian

$$H(t) = \tfrac{1}{2}[p^2 + \omega^2(\epsilon t)q^2] \tag{1}$$

for a particular one-parameter family of analytic functions $\omega(\epsilon t)$, where $1/\epsilon$ stands for the time scale of the system. It is commonly accepted that when the system evolves adiabatically the asymptotic variation of the variable $J = H/\omega$ is exponentially small. Nevertheless, for certain values of the parameter characterizing these frequencies the change in $J(t)$ from $-\infty$ to $+\infty$ exactly vanishes for any value of $\epsilon$. We shall discuss below this paradoxical phenomenon.

For the Hamiltonian in (1) the variation with time of the variable $J$ is a topic frequently discussed in the literature. As is well known, $J$ is an adiabatic invariant which means that its value remains approximately constant during a time interval of order $1/\epsilon$ [4]. A slightly different question arises when considering the variation $\Delta J \equiv J(+\infty) - J(-\infty)$ over the infinite time interval $(-\infty, +\infty)$, instead of the whole history of $J(t)$. It is tacitly supposed that the limiting values $J(\pm\infty)$ exist, which is ensured provided $\omega$ tends sufficiently fast to definite limits as $t \to \pm\infty$. If

$\omega(\epsilon t)$ is an analytical function then $\Delta J \simeq \mathrm{e}^{-k/\epsilon}$, with $k$ a real positive constant and $\epsilon \ll 1$. It is common practice to express this result by saying that the asymptotic variation of the action is exponentially small. Studies on the accuracy in the conservation of $J(t)$ when $\epsilon \ll 1$ have been performed with different techniques, giving generally very elegant results [6–9].

There are, however, potentials for which $\Delta J = 0$, irrespective of whether the regime is sudden or adiabatic. These cases are usually said to correspond to *reflectionless potentials* [2, 3] in analogy with one-dimensional quantum scattering problems. Here we adhere to this terminology. Of course, in the context of classical mechanics neither a reflected nor a transmitted wave is referred to at all.

We are interested in understanding why the variation of the adiabatic invariant vanishes rather than being exponentially small, in the adiabatic regime; and moreover, why the above result holds in fact for any regime. In this note we prove these features for a family of Hamiltonians of the form given in (1).

The particular class of frequencies we study is

$$\omega^2(\epsilon t) = 1 + \frac{\sigma(\sigma-1)\epsilon^2}{\cosh^2 \epsilon t} \tag{2}$$

where $\sigma \geqslant 1$ is a real arbitrary parameter. The frequency considered in [3] corresponds to $\sigma = 2$.

The change of variable

$$z = (1 + \tanh \epsilon t)/2 \tag{3}$$

transforms the equation of motion $\ddot{q} + \omega^2 q = 0$ into Riemann's differential equation

$$q''(z) + \frac{1-2z}{z(1-z)} q' + \left( \frac{1}{4\epsilon^2 z^2 (1-z)^2} + \frac{\sigma(\sigma-1)}{z(1-z)} \right) q = 0 \tag{4}$$

with (regular) singular points $z = 0, 1, \infty$. In the above expressions the dot stands for the time derivative while the prime represents the derivative with respect to $z$. The real solution that presents the adequate behaviour when $\sigma = 1$ is given by

$$q(t) = \alpha \mathrm{e}^{-\mathrm{i}t} {}_2F_1(\sigma, 1-\sigma; 1-\mathrm{i}/\epsilon; z) + \mathrm{CC} \tag{5}$$

in terms of the hypergeometric function ${}_2F_1$. Here $\alpha$ is an arbitrary complex constant to be fixed by initial conditions and CC indicates the complex conjugate of the preceding term. Let us analyse the asymptotic behaviour of $q(t)$, $p(t) = \dot{q}(t)$. When $z \to 0$ (i.e. $t \to -\infty$), then ${}_2F_1 \to 1$ and $q(t)$, $p(t)$ are simply the free HO solutions

$$q(t) \simeq \alpha \mathrm{e}^{\mathrm{i}t} + \mathrm{CC} \qquad p(t) \simeq -\mathrm{i}\alpha \mathrm{e}^{-\mathrm{i}t} + \mathrm{CC}. \tag{6}$$

The free oscillation character of $q(t)$ when $z \to 1$ (i.e. $t \to +\infty$) can be readily seen by taking into account the following analytic continuation [5]:

$$\begin{aligned} {}_2F_1(a, b; c; z) = {} & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} z^{-a} {}_2F_1\left(a, a-c+1; a+b-c+1; 1-\frac{1}{z}\right) \\ & + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} z^{a-c} {}_2F_1\left(c-a, 1-a; c-a-b+1; 1-\frac{1}{z}\right). \end{aligned} \tag{7}$$

After some algebra we get the asymptotic form

$$q(t) \simeq \alpha(\mathrm{e}^{-\mathrm{i}t} R + \mathrm{e}^{\mathrm{i}t} S) + \mathrm{cc} \tag{8}$$

with

$$R = \frac{\Gamma(1-\mathrm{i}/\epsilon)\Gamma(-\mathrm{i}/\epsilon)}{\Gamma(\sigma-\mathrm{i}/\epsilon)\Gamma(1-\sigma-\mathrm{i}/\epsilon)} \qquad S = \frac{\Gamma(1-\mathrm{i}/\epsilon)\Gamma(\mathrm{i}/\epsilon)}{\Gamma(\sigma)\Gamma(1-\sigma)}\,. \tag{9}$$

A similar expression holds true for $p(t)$. Consequently the solution given by (5) may be seen as the product of the asymptotic (oscillatory) solution times a function (${}_2F_1$) describing the finite time corrections to the former. Changes of variables other than (3) may lead to alternative forms of (4) [10]. Nevertheless, the solution no longer necessarily admits such a factorization.

Next we obtain an exact formula for $\Delta J$. From (6) we have $J(-\infty) = 2|\alpha|^2$. To obtain an expression for $J(+\infty)$ from (8) we use the property

$$\Gamma(1-z)\Gamma(z) = \pi/\sin\pi z \tag{10}$$

and a straightforward calculation yields

$$\Delta J = 4|\alpha|^2\rho[\sqrt{1+\rho^2}\cos(2\phi+\xi)+\rho] \tag{11}$$

where $\phi = \arg(\alpha)$, $\rho = |\sin(\pi\sigma)|/\sinh(\pi/\epsilon)$ and

$$\xi = 2[\arg\Gamma(\sigma+\mathrm{i}/\epsilon) + \arg\Gamma(1-\mathrm{i}/\epsilon)] + \tan^{-1}\left(\frac{\tanh\pi/\epsilon}{\tan\pi\sigma}\right). \tag{12}$$

A special situation occurs when $\sigma$ takes on integer values $n > 1$. Then, the above equations give $\Delta J = 0$ irrespective of the $\epsilon$ value. This property is sometimes referred to as reflectionlessness, even in the context of classical mechanics [3]. Furthermore, for these particular frequencies the hypergeometric series in (5) reduces to a Jacobi polynomial [5]

$${}_2F_1(n, 1-n; 1-\mathrm{i}/\epsilon; z) = \frac{(n-1)!}{(1-\mathrm{i}/\epsilon)_{n-1}} P_{n-1}^{(-\mathrm{i}/\epsilon,\mathrm{i}/\epsilon)}(-\tanh\epsilon t)\,. \tag{13}$$

Once (13) is substituted in (5), the exact solution $q(t)$ is expressed in terms of simple algebraic functions. We note in passing that our exact ($\Delta J = 0$) solution coincides with the reflectionless solution obtained *via* the factorization method in soliton theory [11].

Let us now go back to the case of arbitrary $\sigma$ and suppose that $\epsilon \ll 1$, which corresponds to the adiabatic regime. If $\sigma$ is not an integer then $\Delta J$ does not vanish. Since $\omega(\epsilon t)$ is analytic $\Delta J$ must be proportional to $\exp(-k/\epsilon)$ with $k$ some positive real parameter, according to the general rule stated above. In this limiting case we have from (11)

$$\Delta J \simeq 8|\alpha|^2\cos(2\phi+\xi)|\sin(\pi\sigma)|\exp(-\pi/\epsilon)\,. \tag{14}$$

Notice that $\Delta J$ is indeed exponentially small for $\sigma \neq n$ and, what is more interesting, the vanishing of $\Delta J$ stems from the pre-exponential factor. The exponential rule for

$\Delta J$ is still valid for $\sigma$ values neighbouring $\sigma = n$ but right at this point it is the pre-exponential factor in the asymptotic formula that becomes the crucial piece.

In summary, we want to stress that by exactly solving the equation of motion for the one-parameter family of frequencies given by (2) we have found that $\Delta J$ factorizes asymptotically as indicated by (14). The pre-exponential factor depends continuously on the parameter $\sigma$ whereas the exponential itself is a function of $\epsilon$ alone. Just when $\sigma = n > 1$ we have the likely correlated facts that, whatever the regime, $q(t)$ is given by a combination of simple algebraic functions and that $\Delta J = 0$, i.e. reflectionlessness. A similar property has been observed for the structure of the wavefunction of some one-dimensional time-independent quantum systems [2].

It would be interesting to see to what extent these conclusions are valid for other families of frequencies. Needless to say, this might be a poser in view of the scarcity of exactly solvable models.

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