A perturbative algorithm for quasi-periodic linear systems close to constant coefficients

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Abstract

A perturbative procedure is proposed to formally construct analytic solutions for a linear differential equation with quasi-periodic but close to constant coefficients. The scheme constructs the necessary linear transformations involved in the reduction process up to an arbitrary order in the perturbation parameter. It is recursive, can be implemented in any symbolic algebra package and leads to accurate analytic approximations sharing with the exact solution important qualitative properties. This algorithm can be used, in particular, to carry out systematic stability analyses in the parameter space of a given system by considering variational equations.

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1 Introduction

In this paper we consider the differential equation

$$\dot{y} \equiv \frac{dy}{dt} = (A_0 + Q(t,\varepsilon))y, \tag{1}$$

where $y \in \mathbb{C}^d$, $\varepsilon > 0$, A_0 is a constant $d \times d$ matrix and

$$Q(t,\varepsilon) \equiv \sum_{j\geq 1} \varepsilon^j A_j(t) \tag{2}$$

is a quasi-periodic $d \times d$ matrix function of t with frequencies $(\omega_1, \ldots, \omega_r)$.

We recall that a function f is said to be *quasi-periodic* with basic frequencies $\omega = (\omega_1, \ldots, \omega_r)$ if $f(t) = F(\theta_1, \ldots, \theta_r)$, where F is 2π-periodic with respect

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to $\theta_1, \ldots, \theta_r$ and $\theta_j = \omega_j t$ for $j = 1, \ldots, r$. Quasi-periodic functions have a representation of the form

$$f = \sum_{k \in \mathbb{Z}^r} f_k \, \mathrm{e}^{i(k,\omega)t}$$

where $(k, \omega) \equiv k_1\omega_1 + \cdots + k_r\omega_r$ and $\sum |f_k|^2 < \infty$ [10].

In connection with system (1)-(2), the issue of reducibility has received much attention along the years. Roughly speaking, equation (1) is said to be reducible if there exists a change of variables $y = P(t)z$ defined by a nonsingular quasi-periodic and continuously differentiable matrix $P(t)$ such that z satisfies the equation $\dot{z} = Kz$, with K a constant matrix. In the purely periodic case, $A(t + T, \varepsilon) = A(t, \varepsilon)$, with $T > 0$, the well known Floquet theorem [24] guarantees reducibility by means of a periodic transformation $P(t)$ with the same period T. Moreover, every fundamental matrix solution of (1) can be written globally as

$$Y(t) = P(t) \, \mathrm{e}^{Kt}. \tag{3}$$

In the more general case of a quasi-periodic system which is close to constant coefficients, i.e. eq. (2) with sufficiently small ε, the analysis is more involved. At least two different strategies can be found in the literature. The first one, proposed by Shtokalo [23] and later analyzed by several authors [11, 13, 21], consists in formally constructing a change of variables

$$y = P(t, \varepsilon) \, z = \left(I + \sum_{n=1}^{\infty} \varepsilon^n P_n(t) \right) z, \tag{4}$$

as a power series in ε, so that equation (1) with (2) is transformed into

$$\dot{z} = K(\varepsilon) z \equiv \left(A_0 + \sum_{j=1}^{\infty} \varepsilon^j \, K_j \right) z, \tag{5}$$

where K_j are constant matrices. Recursive procedures exist to compute the quasi-periodic matrices $P_n(t)$ and the constant terms K_j at each iteration (see e.g. [6]). Essentially, the K's are determined by averaging and subsequently the P's are obtained by solving the corresponding differential equation. Although the procedure only allows one to construct asymptotic expansions for the solution, it is still possible to provide sufficient conditions guaranteeing stability or instability of the trivial solution of (1) from the solution $z = 0$ of system (5) once truncated at the, say, the sth-iteration [21, 6]. Moreover, the technique can be generalized to analyze the asymptotic behavior of linear differential equations with oscillatory decreasing coefficients [6, 18].

The second approach is very much in the spirit of the proof of KAM theorem in Hamiltonian systems, as given, for instance, in [1], and was first considered in [3]: instead of just one change of variables (4), a sequence of successive quasi-periodic linear transformations is constructed with the aim not of eliminating all powers of ε at once, but to increase in each step the order of the perturbation by the square of the preceding one. In this way, it is possible to establish rigorous

results on the reducibility of (1) when ε is small [3, 15, 14]. More specifically, let λ_i denote the eigenvalues of A_0 and let $\alpha_{ij} = \lambda_i - \lambda_j$ for $i \neq j$. Then if all values $\mathrm{Re}\,\alpha_{ij} \neq 0$ the system is reducible for $|\varepsilon| < \varepsilon_0$, ε_0 sufficiently small [3], whereas if some of the $\mathrm{Re}\,\alpha_{ij}$ are zero (as happens when the λ_i are purely imaginary) more hypotheses are required. Thus, if the α_{ij} and the basic frequencies of $Q(t,\varepsilon)$ satisfy diophantine (non resonant) conditions and a certain non degeneracy holds with respect to ε, then there exists a Cantorian set of positive measure $\mathcal{E}$ such that for $\varepsilon \in \mathcal{E}$ system (1) is reducible by means of a quasi-periodic change of variables [15]. In other words, if the parameter ε is small enough, reducibility can be achieved only for a set of values of ε with empty interior but large Lebesgue measure (maybe full measure). On the other hand, just by assuming a non resonant condition of α_{ij} and the basic frequencies of $Q(t,\varepsilon)$ it is possible to transform the original system (1) by means of a sequence of quasi-periodic matrices to

$$\dot{z} = (K(\varepsilon) + \varepsilon Q^*(t,\varepsilon))z, \qquad |\varepsilon| \leq \varepsilon_0 \tag{6}$$

where Q^* is exponentially small in ε [14]. Equivalently, instead of aiming at a total reduction of system (1), the goal is to minimize the quasi-periodic part (up to exponentially small terms) without taking out any value of ε. In addition, there is no need to impose further non degeneracy conditions [14]. In fact, the procedure developed in [14] allowed the authors to compute numerically for a 2×2 system the change of variables in such a way that, for a given (small) value of ε, the size of the remainder Q^* in (6) is kept below a certain predefined tolerance.

Our purpose in this paper, rather than analyzing conditions guaranteeing total or partial reducibility of system (1), consists in devising an algorithm for constructing the necessary linear transformations involved in the reduction process in such a way that (i) it is computationally well adapted so that the analytic procedure can be carried out at high orders in ε and (ii) the "effective" Floquet factorization that results from the corresponding approximations to the transformation $P(t,\varepsilon)$ and the matrix $K(\varepsilon)$ allows us to construct analytic approximate expressions for the fundamental matrix of (1) in a way that other qualitative properties of the exact solution (e.g., symplecticity or unitarity) are exactly preserved. This is so when only one linear change of variables is involved, as in (4), but also when a sequence of transformations is considered. The algorithm constitutes a generalization of that presented in [8] for periodic systems and consists essentially in constructing $P(t,\varepsilon)$ as a matrix exponential whose generator $L(t,\varepsilon)$ satisfies a cohomological equation at each order. The analytic approximations thus obtained are in addition free of secular terms.

We are well aware that, even when the system is reducible for a given value of ε, the resulting transformation P might be far from the identity. This in fact is quite common for "moderate" values of ε [15, 14, 20]. Under such circumstances, our construction would be purely formal, of course. Nevertheless, as the examples collected in the paper show, the procedure is still able to provide reasonably accurate results even in this situation.

2 The general algorithm

2.1 One transformation

The algorithm we use to construct the approximations can be considered as a generalization of the procedure presented in [8]. For the benefit of the reader, we collect here only the main points of this procedure and refer to [8] for a more detailed treatment. We start with the case of only one transformation.

Let us denote by $Y(t,\varepsilon)$ and $Z(t,\varepsilon)$ the fundamental $d \times d$ matrices of systems (1) and (5), respectively. In other words,

$$\frac{\partial Y}{\partial t} = (A_0 + Q(t,\varepsilon))\, Y(t,\varepsilon), \qquad Y(0,\varepsilon) = I, \tag{7}$$

$$\frac{\partial Z}{\partial t} = K(\varepsilon)\, Z(t,\varepsilon), \qquad Z(0,\varepsilon) = I, \tag{8}$$

so that the corresponding solutions are expressed as

$$y(t) = Y(t,\varepsilon)\, y_0 \qquad \text{and} \qquad z(t) = Z(t,\varepsilon)\, z_0,$$

respectively, in terms of the initial conditions. By means of the transformation $y = P(t,\varepsilon)z$, these fundamental matrices are related by

$$Z(t,\varepsilon) = P^{-1}(t,\varepsilon)\, Y(t,\varepsilon)\, P(0,\varepsilon), \tag{9}$$

whereas the coefficient matrices verify

$$K(\varepsilon) = P^{-1}(t,\varepsilon)(A_0 + Q(t,\varepsilon))P(t,\varepsilon) + \frac{\partial P^{-1}(t,\varepsilon)}{\partial t}\, P(t,\varepsilon). \tag{10}$$

The matrix $P(t,\varepsilon)$ is assumed to satisfy a differential equation similar as (7) but with respect to the perturbation parameter ε, i.e.,

$$\frac{\partial}{\partial \varepsilon} P^{-1}(t,\varepsilon) = L(t,\varepsilon) P^{-1}(t,\varepsilon), \qquad P^{-1}(t,0) = I, \tag{11}$$

where the *generator* $L(t,\varepsilon)$ is a formal power series in ε:

$$L(t,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n L_{n+1}(t). \tag{12}$$

Then the solution of (11) can be expressed as

$$P^{-1}(t,\varepsilon) = \exp \Omega(t,\varepsilon), \qquad \text{where} \qquad \Omega(t,\varepsilon) = \sum_{n=1}^{\infty} \varepsilon^n v_n(t) \tag{13}$$

by applying the Magnus expansion to (11) [17, 2]. The terms $v_n(t)$ are functions of $L_j(t)$ that can be determined recursively [8]. By inserting the corresponding power series in equation (10) and equating terms of the same power in ε one arrives at $K_0 = A_0$ and the so-called cohomological equations

$$\dot{L}_n + [L_n, A_0] = nK_n - F_n, \qquad n \geq 1 \tag{14}$$

with

$$\begin{aligned} F_1 &= A_1 \\ F_n &= \sum_{j=1}^{n-1} [L_{n-j}, K_j] + w_{n-1}, \qquad n \geq 2, \end{aligned} \tag{15}$$

and $w_n = w_n(A_1, \ldots, A_{n+1}, L_1, \ldots, L_n)$ ([8], eq. (24)). In particular,

$$\begin{aligned} &K_0 = A_0 \\ &\dot{L}_1 + [L_1, A_0] = K_1 - A_1 \\ &\dot{L}_2 + [L_2, A_0] = 2K_2 - 2A_2 - [L_1, K_1 + A_1] \\ &\dot{L}_3 + [L_3, A_0] = 3K_3 - 3A_3 - [L_2, K_1 + \frac{1}{2}A_1] - [L_1, K_2 + 2A_2 + \frac{1}{2}[L_1, A_1]]. \end{aligned} \tag{16}$$

Both K_n and $L_n(t)$ are obtained recursively recursively thanks to eq. (16). For later use, we notice that eq. (14) can also be written as

$$\dot{L}_n = \mathrm{ad}_{A_0} L_n + nK_n - F_n \tag{17}$$

in terms of the adjoint operator ad: $\mathrm{ad}_{A_0} B \equiv [A_0, B]$. Once the generators $L_j(t)$, $j = 1, \ldots, m$ have been obtained, we compute the terms $v_1(t), \ldots v_m(t)$ (by means of the recurrence (22) in [8]), the truncated series $\Omega^{[m]}(t, \varepsilon) = \sum_{n=1}^{m} \varepsilon^n v_n(t)$ is formed and finally the transformation $P(t, \varepsilon)$ is determined as $P(t, \varepsilon) = \exp(-\Omega^{[m]}(t, \varepsilon))$.

Of course, it is also possible to construct directly the transformation as (13). In that case, by substituting this series into (10), an equation of type (14) is obtained for the terms $v_n(t)$, although the structure of the functions appearing in the equation is more tricky.

2.2 Determining the generator

Obtaining the generators $L_n(t)$ is a crucial step according to our objectives. This latter operation must be done at each step by solving the differential equation (14) with the requirement that (i) K_n is constant and (ii) $L_n(t)$ is a quasi-periodic function with the same frequencies as $Q(t, \varepsilon)$.

Several possibilities exist for determining $L_n(t)$ verifying such requirements (see e.g. [3, 15]). Here, as in [9, 8], we write the formal solution of equation (17) as

$$L_n(t) = \mathrm{e}^{t\,\mathrm{ad}_{A_0}} L_n(0) + \mathrm{e}^{t\,\mathrm{ad}_{A_0}} \int_0^t \mathrm{e}^{-s\,\mathrm{ad}_{A_0}} (nK_n - F_n(s))\, ds, \tag{18}$$

where

$$\mathrm{e}^{t\,\mathrm{ad}_{A_0}} B = \mathrm{e}^{tA_0} B \mathrm{e}^{-tA_0}$$

for any matrix B. Here, by induction, $F_n(t)$ is a quasi-periodic matrix function with basic frequencies $(\omega_1, \ldots, \omega_r)$, so that

$$F_n(t) = \sum_{k \in \mathbb{Z}^r} C_{n,k}\, \mathrm{e}^{i(k,\omega)t}. \tag{19}$$

We then take $nK_n = \langle F_n \rangle$, where $\langle F_n \rangle$ denotes the limiting mean value of $F_n(t)$, i.e.,

$$\langle F_n \rangle \equiv \lim_{T\to\infty} \frac{1}{T} \int_a^{a+T} F_n(t)dt, \tag{20}$$

and this limit does not depend upon the choice of a in (20), as shown in [10]. In consequence,

$$nK_n - F_n(t) = \langle F_n \rangle - F_n(t) = -\sum_{k\in\mathbb{Z}^r\setminus\{0\}} C_{n,k}\, \mathrm{e}^{i(k,\omega)t}.$$

Assume for the time being that the linear system consisting of d^2 equations

$$\Big(\mathrm{ad}_{A_0} - i(k,\omega)I\Big)X = C_{n,j} \tag{21}$$

has a unique solution, which we denote as $X \equiv R_{n,j}$. Then we can write

$$\begin{aligned} G_n(t) &\equiv \int \mathrm{e}^{-t\,\mathrm{ad}_{A_0}} (\langle F_n \rangle - F_n(t))\, dt = -\int dt \sum_{k\in\mathbb{Z}^r\setminus\{0\}} \mathrm{e}^{-t\,\mathrm{ad}_{A_0}} C_{n,k}\, \mathrm{e}^{i(k,\omega)t} \\ &= \int dt \sum_{k\in\mathbb{Z}^r\setminus\{0\}} \mathrm{e}^{-t\,\mathrm{ad}_{A_0}+i(k,\omega)tI}(-\mathrm{ad}_{A_0} + i(k,\omega)I)R_{n,k} \\ &= \sum_{k\in\mathbb{Z}^r\setminus\{0\}} \mathrm{e}^{-t\,\mathrm{ad}_{A_0}+i(k,\omega)tI} R_{n,k} = \mathrm{e}^{-t\,\mathrm{ad}_{A_0}} \sum_{k\in\mathbb{Z}^r\setminus\{0\}} R_{n,k}\, \mathrm{e}^{i(k,\omega)t}, \end{aligned} \tag{22}$$

and thus eq. (18) leads to

$$\begin{aligned} L_n(t) &= \mathrm{e}^{t\,\mathrm{ad}_{A_0}}\, (L_n(0) - G_n(0)) + \mathrm{e}^{t\,\mathrm{ad}_{A_0}} G_n(t) \\ &= \mathrm{e}^{t\,\mathrm{ad}_{A_0}}\, (L_n(0) - G_n(0)) + \sum_{k\in\mathbb{Z}^r\setminus\{0\}} R_{n,k}\, \mathrm{e}^{i(k,\omega)t}, \end{aligned}$$

so that if we choose $L_n(0) = G_n(0)$ we then have

$$L_n(t) = \sum_{k\in\mathbb{Z}^r\setminus\{0\}} R_{n,k}\, \mathrm{e}^{i(k,\omega)t} \tag{23}$$

i.e., the generator $L_n(t)$ is a quasi-periodic function with the same basic frequencies as $F_n(t)$.

Let us now analyze under which conditions the matrix system of algebraic equations (21) admits a unique solution $R_{n,j}$. This is fulfilled if and only if $X = 0$ is the unique solution of

$$\Big(\mathrm{ad}_{A_0} - i(k,\omega)I\Big)X = A_0X - X(A_0 + i(k,\omega)I) = 0. \tag{24}$$

Let $\{\lambda_j\}_{j=1}^s$ denote the distinct eigenvalues of A_0. Then the eigenvalues of $A_0+i(k,\omega)I$ are $\{\lambda_j+i(k,\omega)\}_{j=1}^s$. If $\lambda_\ell-\lambda_m-i(k,\omega) \neq 0$ for all $\ell, m \in \{1,\ldots,s\}$ then A_0 and $A_0 + i(k,\omega)I$ do not have common characteristic values, so that, according to [12, p. 220], equation (24) admits $X = 0$ as unique solution. As usual when dealing with this kind of equations, and in order to avoid the

problem of small divisors, a diophantine condition is introduced, namely we assume that

$$|\lambda_\ell - \lambda_m - i(k,\omega)| > \frac{\delta}{|k|^\gamma} \qquad \forall\, \ell, m \in \{1,\dots,s\} \quad \forall\, k \in \mathbb{Z}^r \backslash \{0\} \tag{25}$$

for some constants $\delta > 0$ and $\gamma > r-1$, see [14, 4, 5] for more details. Here $|k| = |k_1| + \cdots + |k_r|$ This condition avoids the presence of small denominators in the expression (23) for $L_n(t)$.

We have thus proved the following result.

Proposition 1 *Consider the matrix equation* $\dot{L}_n(t) = [A_0, L_n(t)] + nK_n - F_n(t)$, $n \geq 1$, *where* A_0 *is a constant matrix with distinct eigenvalues* $\lambda_1, \dots, \lambda_s$, $F_n(t)$ *is a quasi-periodic matrix with basic frequencies* $\omega_1, \dots, \omega_r$ *and* K_n *is a constant matrix to be determined. If we assume the diophantine condition (25) and fix*

$$K_n = \frac{1}{n}\langle F_n \rangle, \tag{26}$$

where $\langle F_n \rangle$ *denotes the mean value (20), then there exists a unique solution* $L_n(t)$ *with the same frequencies as* $F_n(t)$.

As a matter of fact, this unique solution $L(t)$ can be obtained by two different but equivalent procedures: either by computing directly the antiderivative $G_n(t)$ in (22) and then

$$L_n(t) = \mathrm{e}^{t\,\mathrm{ad}_{A_0}} G_n(t) = \mathrm{e}^{tA_0} G_n(t) \mathrm{e}^{-tA_0} \tag{27}$$

or by computing first the expansion (19) and then forming equation (23) for each $k \in \mathbb{Z}^r \backslash \{0\}$, where the matrix $R_{n,k}$ is the unique solution of system (21). In both cases we have $L_n(0) = G_n(0)$ and $\langle L_n \rangle = 0$.

This process can be carried out for any $n \geq 1$, so that after m steps the fundamental matrix $Y(t,\varepsilon)$ can be formally written as

$$Y(t,\varepsilon) \approx \mathrm{e}^{-\Omega^{[m]}(t,\varepsilon)}\, \mathrm{e}^{tK^{[m]}(\varepsilon)}\, \mathrm{e}^{\Omega^{[m]}(0,\varepsilon)}, \tag{28}$$

where

$$\Omega^{[m]}(t,\varepsilon) = \sum_{n=1}^{m} \varepsilon^n v_n(t), \qquad K^{[m]}(\varepsilon) = A_0 + \sum_{n=1}^{m} \varepsilon^n K_n \tag{29}$$

and $v_n(t)$ are determined from the generators $L_j(t)$ (e.g., by recursion (22) in [8]).

A remarkable property of equation (17) is the following. If we have constructed one solution K_n and $L_n(t)$, then we can get infinite solutions of the form

$$\tilde{K}_n = K_n - (1/n)[A_0, U], \qquad \tilde{L}_n(t) = L_n(t) + U,$$

where U is an arbitrary constant matrix. This can be verified just by substituting the expressions of $\tilde{K}_n$ and $\tilde{L}_n(t)$ in (17). If we choose $U = -G_n(0)$ then $\tilde{L}_n(0) = 0$ for all n and the solution reads

$$\begin{aligned} \tilde{K}_n &= (1/n)(\langle F_n \rangle + [A_0, G_n(0)]) \\ \tilde{L}_n(t) &= -G_n(0) + \mathrm{e}^{t\,\mathrm{ad}_{A_0}} G_n(t). \end{aligned} \tag{30}$$

In consequence, we get the approximation

$$Y(t,\varepsilon) \approx \mathrm{e}^{-\tilde{\Omega}^{[m]}(t,\varepsilon)}\, \mathrm{e}^{t\tilde{K}^{[m]}(\varepsilon)} \tag{31}$$

where, as before,

$$\tilde{\Omega}^{[m]}(t,\varepsilon) = \sum_{n=1}^{m} \varepsilon^n \tilde{v}_n(t), \qquad \tilde{K}^{[m]}(\varepsilon) = \sum_{n=1}^{m} \varepsilon^n \tilde{K}_n \tag{32}$$

but $\tilde{v}_n(0) = 0$ for all n. Notice that, although (28) and (31) have the same properties concerning the stability (in other words, both $K^{[m]}$ and $\tilde{K}^{[m]}$ have the same eigenvalues up to ε^m), they render in fact different approximations to the exact solution.

2.3 Illustrative examples

We next illustrate the main features of the previous algorithm by applying it to two simple examples. In the first example we also analyze how results on the stability of the system can be derived from the procedure.

Example 1: generalized Hill's equation. Assume we have a pendulum with friction which may rotate in a plane around a point of suspension which can itself undergo vertical oscillations. Then, the equation of motion of the system near the upper position of equilibrium is given by [23]

$$\ddot{\varphi} + \varepsilon c\, \dot{\varphi} + \varepsilon(q(t) - \varepsilon\rho^2)\, \varphi = 0, \tag{33}$$

where φ stands for the small deviation from the upper equilibrium, $\varepsilon > 0$, $c > 0$ accounts for the friction in the system, ρ depends on the length of the pendulum and gravity, and

$$q(t) = \sum_{k=1}^{r} b_k \cos(\omega_k t)$$

is a trigonometric polynomial accounting for the oscillations of the suspension point. Equation (33) can be replaced by

$$\begin{aligned} \frac{dy_1}{dt} &= y_2 \\ \frac{dy_2}{dt} &= -\varepsilon\, c\, y_2 - \varepsilon(q(t) - \varepsilon\rho^2) y_1 \end{aligned}$$

or in matrix form

$$\frac{dy}{dt} = (A_0 + \varepsilon A_1(t) + \varepsilon^2 A_2(t))\, y \tag{34}$$

with

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & 0 \\ -q(t) & -c \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 \\ \rho^2 & 0 \end{pmatrix}.$$

In this case $\lambda_{1,2} = 0$, so that the diophantine condition (25) writes $|(k,\omega)| \geq \delta/|k|^\gamma$. K_n and $L_n(t)$ are computed following (26) and (27), respectively, so that up to second order ($K_0 = A_0$) we get

$$K_1 = \begin{pmatrix} 0 & 0 \\ 0 & -c \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 0 & 0 \\ \rho^2 - q_0 & 0 \end{pmatrix}, \qquad \text{with} \qquad q_0 = \sum_{k=1}^{r} \frac{b_k^2}{2\omega_k^2},$$

whereas the expressions for the generators $L_1(t)$, $L_2(t)$ are more involved. When looking at the stability of the trivial solution of equation (34) for sufficiently small ε, then the eigenvalues of the new matrix $K^{[m]}(\varepsilon)$ given in (29) are to be computed, or equivalently the trace and determinant of $K^{[m]}(\varepsilon)$. Up to second order, these are

$$\mathrm{tr} K^{[2]} = -\varepsilon\, c, \qquad \det K^{[2]} = -\varepsilon^2(\rho^2 - q_0).$$

Since $K^{[2]}$ has a negative trace, the eigenvalues would have negative real parts if the determinant is positive, i.e., if $\rho^2 - q_0 < 0$, or

$$-\rho^2 + \sum_{k=1}^{r} \frac{b_k^2}{2\omega_k^2} > 0. \tag{35}$$

Therefore, the asymptotic stability follows if (35) holds. Carrying out the process up to order 3 we get analogously

$$K_3 = c\, q_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \text{with} \qquad q_1 = \sum_{k=1}^{r} \frac{b_k^2}{3\omega_k^4}.$$

The trace of $K^{[3]}$ takes the same value as before, whereas

$$\det K^{[3]} = -\varepsilon^2\big(\varepsilon^2 c^2 q_1 + \varepsilon^4 c^2 q_1^2 + (\rho^2 - q_0)\big).$$

In consequence, the condition for asymptotic stability is now

$$\varepsilon^2 c^2 q_1 + \varepsilon^4 c^2 q_1^2 + (\rho^2 - q_0) < 0$$

and this is true for sufficiently small ε if (35) holds. In general, one finds for the successive K_j the following pattern:

$$K_{2n} = \begin{pmatrix} 0 & \beta_{2n,2} \\ \beta_{2n,3} & 0 \end{pmatrix}, \qquad K_{2n+1} = \begin{pmatrix} \beta_{2n+1,1} & 0 \\ 0 & \beta_{2n+1,4} \end{pmatrix}, \qquad n = 1, 2, 3, \ldots,$$

where $\beta_{\ell,j}$ are (complicated) functions of the parameters of the problem and $\beta_{2,2} = 0$. In any case, the same conclusion is achieved: a necessary condition for asymptotic stability for sufficiently small ε is eq. (35). This result is in agreement with the analysis carried out by standard averaging theory [23, 6].

If we apply equations (30)-(32) instead of the former, we clearly get different expressions for the new coefficient matrices at each order. Thus, in particular,

$$\tilde{K}_1 = \begin{pmatrix} 0 & 2q_2 \\ 0 & -c \end{pmatrix}, \qquad \text{with} \qquad q_2 = \sum_{k=1}^{r} \frac{b_k}{\omega_k^2}.$$

In any case, both $\tilde{K}^{[m]}$ and $K^{[m]}$ have the same eigenvalues up to the corresponding order [8].

To have a more quantitative description, we next consider a particular set of coefficients in (34), namely

$$\left(c = 1, r = 2, b_1 = 2, b_2 = 1, \omega_1 = \sqrt{2}, \omega_2 = 1, \rho = 1\right). \tag{36}$$

We have carried out the two variants (26)-(29) and (30)-(32) of the procedure proposed in section 2.2 and computed the corresponding approximations (28) and (31) to the fundamental matrix. These approximations are compared with the results achieved by applying high order averaging (i.e., equations (4)-(5)) up to the same order in the parameter ε as done e.g. in [6].

In Figure 1 we show the error (in logarithmic scale) of the corresponding approximation with $m = 2, 6$ with respect to the exact solution (computed by numerical integration up to high precision) for two values of the perturbation parameter: $\varepsilon = 0.05$ and $\varepsilon = 0.1$. The error is computed as the Frobenius norm of the difference between the exact and the approximate fundamental matrix.

To discern how this error behaves for larger time intervals, in Figure 2 we depict the results achieved by the same approximations up to ε^6 in the interval $t \in [60, 120]$. Notice that the procedure (26)-(29) provide the most accurate results in all cases and the improvement over the standard high order averaging method is quite remarkable, especially for small values of ε.

Example 2: a periodic system. The formalism of section 2.1 can also be applied of course to a periodic system. We consider now the Schrödinger equation ($\hbar = 1$) for the evolution operator $U(t)$,

$$i\,\dot{U}(t) = H(t)\,U(t), \qquad U(0) = I, \tag{37}$$

with Hamiltonian

$$H(t) = \frac{1}{2}\omega_0\,\sigma_3 + \varepsilon\left(\sigma_1 \cos\omega t + \sigma_2 \sin\omega t\right). \tag{38}$$

Here σ_i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and ε, ω_0, $\omega \neq \omega_0$ are real parameters. The exact solution of (37) is then

$$U_{\rm ex}(t) = \exp\left(\frac{-i}{2}\omega t \sigma_3\right) \exp\left[-it\left(\frac{1}{2}(\omega_0 - \omega)\sigma_3 + \varepsilon\sigma_1\right)\right], \tag{39}$$

and the matrix $U(t)$ is unitary: $U(t)U^\dagger(t) = U^\dagger(t)U(t) = I$, where $U^\dagger$ denotes the conjugate transpose of U. This feature has the important physical consequence that transition probabilities between different quantum states remain in the interval $[0, 1]$ for all t.

Notice that equation (37) can be recast in the form (7) with

$$A_0 = -\frac{i}{2}\omega_0\sigma_3, \qquad \frac{1}{\varepsilon}Q(t,\varepsilon) = A_1(t) = -i\left(\sigma_1 \cos\omega t + \sigma_2 \sin\omega t\right)$$

with the eigenvalues of A_0 being purely imaginary. Once the recursive procedure (26)-(27) is computed for several orders the general term of K_n and $L_n(t)$ is guessed by inspection. Specifically, we get

$$K_{2j-1} = 0, \qquad K_{2j} = \frac{(-1)^{j+1} i \sigma_3 C_{j-1}}{(\omega - \omega_0)^{2j-1}}, \qquad j = 1, 2, \ldots$$

where C_j denote the Catalan numbers [19]. The series $\sum_{j\geq 1} \varepsilon^j K_j$ can be summed up in closed form resulting in

$$K(\varepsilon) = \frac{i}{2}\left(\sqrt{(\omega-\omega_0)^2 + 4\varepsilon^2} - \omega\right)\sigma_3. \tag{40}$$

On the other hand, for the generators we get

$$L_{2j} = 0, \qquad L_{2j-1}(t) = \frac{(-1)^j\, i\, 2^{2j-2}}{(\omega-\omega_0)^{2j-1}}(\cos\omega t\sigma_2 - \sin\omega t\sigma_1)$$

$j = 1, 2, \ldots$, so that

$$L(t,\varepsilon) = \frac{i(\omega-\omega_0)}{4\varepsilon^2}(\cos\omega t\sigma_2 - \sin\omega t\sigma_1)\sum_{j=1}^{\infty}(-1)^j\left(\frac{2\varepsilon}{\omega-\omega_0}\right)^{2j}. \tag{41}$$

The series in (41) corresponds to the function

$$L(t,\varepsilon) = \frac{i(\omega-\omega_0)}{(\omega-\omega_0)^2 + 4\varepsilon^2}\left(\sin\omega t\sigma_1 - \cos\omega t\sigma_2\right) \tag{42}$$

in the domain

$$|\omega - \omega_0| > 2\varepsilon. \tag{43}$$

Notice that in this case condition (25) reads $|\omega_0 + k\omega| > \delta/|k|$, $k \in \mathbb{Z}\backslash\{0\}$, but in fact the solution for the generator provided by the procedure is valid in the larger domain (43). The actual transformation is obtained as $P(t,\varepsilon) = \exp(-\Omega(t,\varepsilon))$, where the exponent can be determined (up to an arbitrary order) either by computing the previous recurrences for the terms $v_i(t)$, or by applying directly the Magnus expansion to equation (11) with L given by (41) [2].

To illustrate the behavior of the approximations in a particular situation, we fix $\omega_0 = 1$ and $\omega = 3$, so that the previous algorithm for fixing K and the generator L converges if $\varepsilon < 1$. In Figure 3 we show in logarithmic scale the error in the probability of transition between states 1 and 2, $|(U_{\rm ex}(t))_{12}|^2 - |(U_{\rm approx}(t))_{12}|^2$, obtained when the approximate evolution operator $U_{\rm approx}(t)$ is computed up to ε^7 (dashed line) and ε^{15} (solid line) with (40)-(42) for $\varepsilon = 0.5$. Notice how including additional terms in the expansions provide more accurate results (the value of ε belongs to the domain of convergence) and the approximations do not substantially deteriorate even if one consider long time intervals. It is worth stressing that these approximations are unitary by construction.

3 Successive transformations

The algorithm developed in section 2 aims at constructing one transformation $P(t,\varepsilon)$ in such a way that in the new coordinates all the time dependency in the coefficient matrix of equation (1) is (at least formally) removed up to a certain order in the perturbation parameter ε whilst the corresponding approximate solutions thus obtained still share with the exact solution of the problem some qualitative properties. The procedure does not converge in general, although for the previous Example 2 it provides indeed unitary convergent approximations if the parameters satisfy condition (43).

Much in the same spirit as in KAM methods for Hamiltonian mechanics, one could consider a different but related alternative, namely an iteration of the previous procedure where now in each step one uses a new (and in a sense 'improved') unperturbed matrix. Specifically, in equations (16) one calculates the L's simultaneously to as many orders as one can by choosing the K's in the same way as before, but without coupling L's of different orders in the same equation. In the remaining orders, the L's are set equal to zero and the equations are then used to find the corresponding K's. This fact is just one variant among many other possibilities along the same lines explored in the literature (e.g. [14, 16, 22, 7]).

For the sake of illustration, we next carry out the first two iterations and refer the reader to e.g. [16] for a more detailed discussion. In the first step we construct a transformation $P_1(t,\varepsilon) = \exp(-\Omega_1(t,\varepsilon))$ generated by $L^{(1)}(t,\varepsilon)$ and leading to a new coefficient matrix $K^{(1)}(\varepsilon)$. We then must deal with equations (17). Analogously, we set

$$K_0^{(1)} = A_0, \qquad K_1^{(1)} = \langle A_1 \rangle,$$

whereas $L_1^{(1)}(t)$ is determined as

$$L_1^{(1)}(t) = \mathrm{e}^{t\,\mathrm{ad}_{A_0}} G_1^{(1)}(t), \qquad \text{with} \quad G_1^{(1)}(t) = \int \mathrm{e}^{-t\,\mathrm{ad}_{A_0}} (K_1^{(1)} - A_1(t))\, dt$$

and $L_n^{(1)}(t) = 0$ for $n > 1$ (since $L_1^{(1)}(t)$ is already present in the equations for $L_2^{(1)}(t), L_3^{(1)}(t), \dots$). Then, for $n > 1$, equation (17) reads $0 = nK_n^{(1)} - F_n^{(1)}(t)$, so that

$$K_n^{(1)} = \frac{1}{n} F_n^{(1)} = \frac{1}{n}([L_1^{(1)}, K_{n-1}^{(1)}] + w_{n-1}^{(1)}),$$

but now $w_{n-1}^{(1)}$ only depends on A_j and $L_1^{(1)}(t)$ (already known). In this way one computes $K_2^{(1)}, K_3^{(1)}, \dots$ which now depend on time. We arrive finally at

$$K^{(1)}(t) = K_0^{(1)} + \varepsilon K_1^{(1)} + \sum_{j\geq 2} \varepsilon^j K_j^{(1)}(t),$$

where the piece $K_0^{[1]}(\varepsilon) \equiv K_0^{(1)} + \varepsilon K_1^{(1)}$ is constant and $K_j^{(1)}(t)$ $(j \geq 2)$ are quasi-periodic functions of t. We are now in position to start the second iteration, i.e., to construct a new transformation $P_2(t,\varepsilon) = \exp(-\Omega_2(t,\varepsilon))$ generated by

$L^{(2)}(t,\varepsilon)$ and coefficient matrix $K^{(2)}(\varepsilon)$, but with the new 'unperturbed' matrix $K_0^{[1]}(\varepsilon)$. For the resulting coefficient matrix $K^{(2)}(\varepsilon) = \sum_{j\geq 0} \varepsilon^j K_j^{(2)}$ after completing the second step we propose

$$K_0^{(2)}(\varepsilon) = K_0^{(1)} + \varepsilon K_1^{(1)}, \qquad K_1^{(2)} = 0, \qquad K_2^{(2)} = \frac{1}{2}\langle F_2^{(2)}\rangle, \qquad K_3^{(2)} = \frac{1}{3}\langle F_3^{(2)}\rangle,$$

and $K_j^{(2)} = \frac{1}{j} F_j^{(2)}$ for $j \geq 4$, whereas for the corresponding generator one has

$$L_1^{(2)} = 0, \qquad L_2^{(2)} = \mathrm{e}^{t\,K_0^{[1]}} G_2^{(2)}(t)\, \mathrm{e}^{-t\,K_0^{[1]}}, \qquad L_3^{(2)} = \mathrm{e}^{t\,K_0^{[1]}} G_3^{(2)}(t)\, \mathrm{e}^{-t\,K_0^{[1]}}$$

and $L_j^{(2)} = 0$ for $j \geq 4$. In this way,

$$K^{(2)}(t,\varepsilon) = K_0^{[2]}(\varepsilon) + \varepsilon^4 K_4^{(2)}(t) + \varepsilon^5 K_5^{(2)}(t) + \cdots,$$

where

$$K_0^{[2]}(\varepsilon) = K_0^{[1]} + \varepsilon^2 K_2^{(2)} + \varepsilon^3 K_3^{(2)}$$

is independent of time, and $K_j^{(2)}(t)$, $j \geq 4$ are quasi periodic. The fundamental matrix of eq. (1) is accordingly factorized as

$$Y(t,\varepsilon) = P_1(t,\varepsilon)\, P_2(t,\varepsilon)\, Z_2(t,\varepsilon)\, P_2^{-1}(0,\varepsilon)\, P_1^{-1}(0,\varepsilon),$$

where $P_i(t,\varepsilon) = \exp(-\Omega_i(t,\varepsilon))$ and Z_2 obeys the differential equation $\dot{Z}_2 = K^{(2)}(t,\varepsilon) Z_2$.

Proceeding in the same way, at the third step one can eliminate the time dependency at orders four through seven, etc. [16]. After n iterations one has

$$Y(t,\varepsilon) = P_1(t,\varepsilon)\cdots P_n(t,\varepsilon)\, Z_n(t,\varepsilon)\, P_n^{-1}(0,\varepsilon)\cdots P_1^{-1}(0,\varepsilon) \tag{44}$$

with

$$\dot{Z}_n = K^{(n)}(t,\varepsilon) Z_n, \qquad K^{(n)}(t,\varepsilon) = K_0^{[n]}(\varepsilon) + \varepsilon^{2^n} K_{2^n}^{[n]}(\varepsilon) + \cdots. \tag{45}$$

If one decides to truncate at this point, then we take $K^{(n)}(t,\varepsilon) = K_0^{[n]}(\varepsilon)$ in (45), $Z_n(t,\varepsilon)$ is approximated by $\exp(tK_0^{[n]}(\varepsilon))$ and, for consistency,

$$P_j(t,\varepsilon) \approx \tilde{P}_j(t,\varepsilon) \equiv \exp\big(-\Omega_j^{[2^n-1]}(t,\varepsilon)\big), \quad \text{with} \quad \Omega_j^{[m]}(t,\varepsilon) = \sum_{r=1}^{m} \varepsilon^r v_{j,r}.$$

In this way the approximation is constructed as

$$Y(t,\varepsilon) = \tilde{P}_1(t,\varepsilon)\cdots\tilde{P}_n(t,\varepsilon)\, \mathrm{e}^{tK_0^{[n]}(\varepsilon)}\, \tilde{P}_n^{-1}(0,\varepsilon)\cdots\tilde{P}_1^{-1}(0,\varepsilon). \tag{46}$$

At each step of the procedure we have to deal with equations of the type (17), and so, according with Proposition 1, a condition like (25) is necessary, but now the quantities $\lambda_\ell(\varepsilon) - \lambda_m(\varepsilon)$ clearly depend on the perturbation parameter. Even in this case, additional conditions can be imposed so that either it is possible to cancel all of the harmonics below a certain threshold in the Fourier expansion of $K^{(n)}(t,\varepsilon)$ [14] or vanish all of them in a Cantorian set [15].

Rather than analyzing the convergence of the procedure we next illustrate how it behaves in practice when the approximation (46) is considered for some examples.

Example 2 revisited. As a first illustration we take again the simple periodic example (37)-(38) with exact solution (39) and construct up to four successive transformations by following the previous iterative scheme. To compare with the procedure of section 2.2, we also include the result achieved by applying the procedure (28)-(29). As before, we fix $\omega_0 = 1$, $\omega = 3$ but now $\varepsilon = 1.02$, so that we are outside the domain of convergence given by (43). In Figure 4 we depict the corresponding error in the approximation obtained after three transformations (so that the approximation is correct up to ε^7, dotted red line) and four linear changes of coordinates (approximation of order ε^{15}, solid line), together with the error corresponding to the approximation of order ε^{15} obtained by truncating the series (40) and (42) (dashed blue line). Notice that the new iterative scheme provides more accurate results in this situation.

Example 3. As a final example, we consider again equation (37), but now with a quasi periodic Hamiltonian:

$$H(t) = \frac{1}{2}\omega_0\,\sigma_3 + \varepsilon\,(\cos t + \cos \omega t)\;\sigma_1. \tag{47}$$

This system describes the dynamics of spin-$\frac{1}{2}$ particles in an oscillating magnetic field [22]. We take $\omega_0 = 2$, $\omega = \sqrt{3}$ and compute three successive linear transformations by following the previous algorithm, i.e., approximation (46) up to $n = 3$. As before, we compare with the procedure (28)-(29) involving only one transformation. In principle, with three transformations it is possible to construct an approximation which is correct up to order ε^7 when all steps in the previous algorithm are carried out. For the sake of illustration, here we also compare with the result obtained when $\tilde{P}_1$, $\tilde{P}_2$ in (46) are computed up to order ε^7, but $\tilde{P}_3 = \exp(-\Omega_j^{[5]})$ so as to speed up the calculation. The corresponding errors in the probability of transition between states 1 and 2 are depicted in Figure 5 as a function of time for $\varepsilon = 0.6$, in the interval $t \in [0, 10]$ (left) and $t \in [100, 110]$ (right), but with the same initial condition $U(0) = I$. Solid lines correspond to the result achieved by after carrying out three complete transformations (so that the approximation is correct up to ε^7), dotted lines are obtained with three transformations consistent up to order ε^5, whereas dashed lines correspond to the algorithm (28)-(29) up to ε^7. We see that even when the last transformation $\tilde{P}_3$ is not computed up to the prescribed order, it is still possible to obtain more accurate results with the new procedure, and that this accuracy does not deteriorate with time. We also should emphasize that both approximations are unitary by construction, just as the physical system they mimic.

As is well known, for a given ω_0, there exist in the parameter space (ω, ε) the so-called resonance tongues. In fact, these resonance phenomena take place in an open and dense subset union of tongues, whereas in its complement there exists a nowhere dense set of positive measure, corresponding to multi- or quasi-periodic dynamics (see e.g. [5] and references therein for an overview of the theory). Our algorithm may in principle be used to analyze these regions in the parameter space and characterize in particular the stability regions and

transition curves separating different kinds of motion. As a matter of fact, in [8] it was used to perturbatively determine the first transition curve in the Mathieu equation. In contrast with numerical procedures (as in e.g. [25]), this algorithm is able to obtain approximate analytic expressions so that it may simplify the analysis in the parameter space.

4 Concluding remarks

In this paper we have presented a perturbative algorithm based on linear transformations that leads to constructing analytic approximations to the fundamental matrix of a linear system with quasi periodic coefficients in the form of a Floquet type factorization. This can be done either by carrying out just one or a sequence of transformations. The very structure of the factorization guarantees that important qualitative properties of the system (such as symplecticity or unitarity) are preserved by the approximations, which are also free from secular terms. As a result, as the examples show, they provide reasonably accurate results even for large time intervals. This technique may also be used as a convenient tool to analyze in practice the reducibility of the system and its stability properties in the parameter space, since it can be easily implemented in symbolic algebra packages.

The convergence of the procedure has not been analyzed here. As in other related procedures, we can say that only asymptotic expansions are expected when one transformation is carried out, whereas, by applying the same techniques as in [3, 14, 15] it is possible in principle to establish convergence for a range of values of ε when enough successive transformations are involved. In any case, as Example 2 above shows, convergence can be guaranteed for particular problems.

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List of Figures

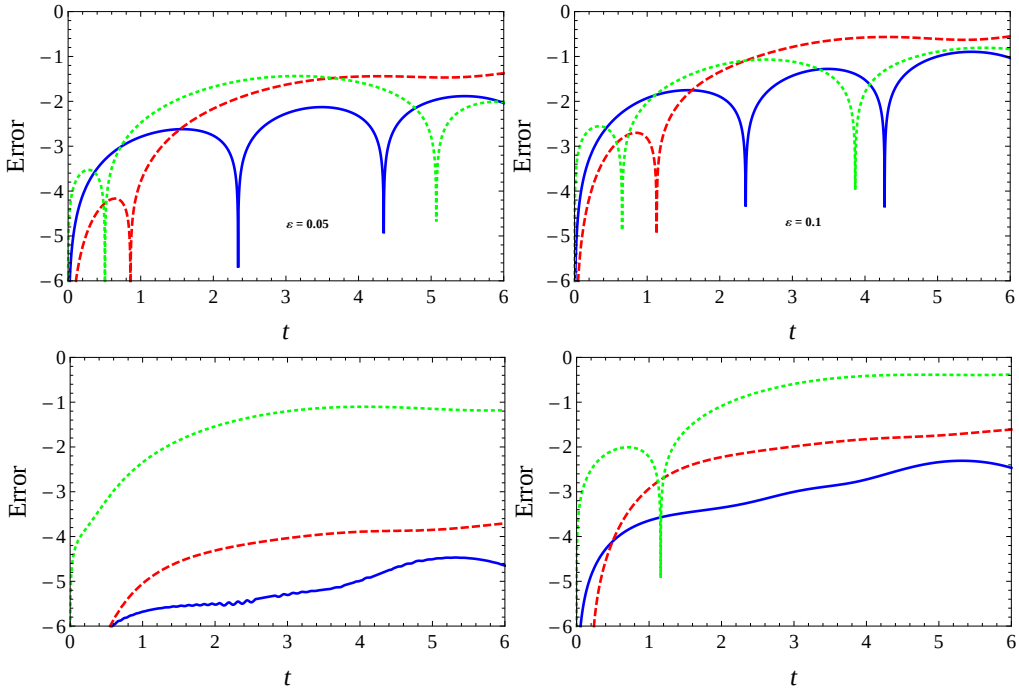
1 Error (in logarithmic scale) of the approximation of order ε^2 (top diagrams) and ε^6 (bottom) with respect to the exact solution for the generalized Hill equation. Solid (blue) line corresponds to scheme (28), dashed (red) line stands for (31) and dotted (green) line is obtained by high-order averaging. Left column corresponds to $\varepsilon = 0.05$ and right column to $\varepsilon = 0.1$. 19

2 Same diagram as in Figure 1 bottom, but now in the interval $t \in [60, 120]$: error of the approximation of order ε^6. Solid (blue) line corresponds to scheme (28), dashed (red) line stands for (31) and dotted (green) line is obtained by high-order averaging. 20

3 Error in the transition probability (in logarithmic scale) of the approximation of order ε^7 (dashed line), and ε^{15} (solid line) for the periodic example (37)-(38) obtained with algorithm (28)-(29). Two different time intervals are considered for the same integration: $t \in [0, 10]$ (left) and $t \in [90, 100]$ (right). 21

4 Error in logarithmic scale with respect to the exact solution. Solid (black) line and dotted (red) line correspond to the approximations computed after successive transformations of orders ε^{15} and ε^7, respectively and dashed (blue) line stands for an approximation of order ε^{15} with scheme (28)-(29). Here $\omega_0 = 1$, $\omega = 3$ and $\varepsilon = 1.02$. 22

5 Error in logarithmic scale with three successive transformations up to order ε^7 (solid line), up to order ε^5 (dotted line) and scheme (28)-(29) (dashed line) up to order ε^7 for the Hamiltonian (47). The parameters are $\omega_0 = 2$, $\omega = \sqrt{3}$ and $\varepsilon = 0.6$. 23



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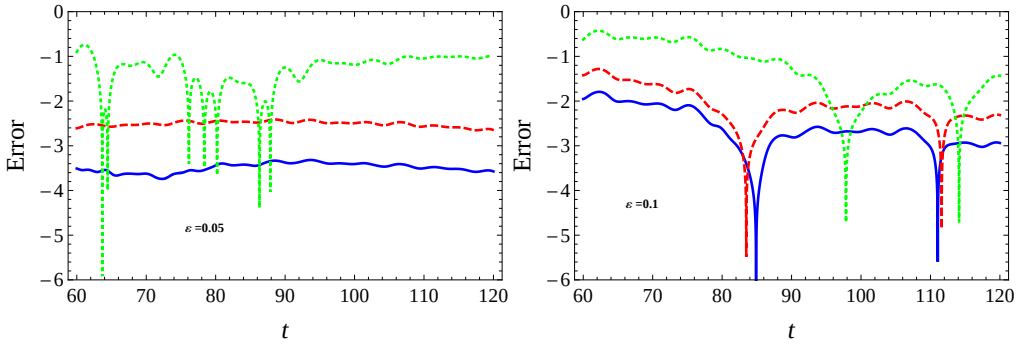


Figure 2: Same diagram as in Figure 1 bottom, but now in the interval $t \in [60, 120]$: error of the approximation of order ε^6. Solid (blue) line corresponds to scheme (28), dashed (red) line stands for (31) and dotted (green) line is obtained by high-order averaging.

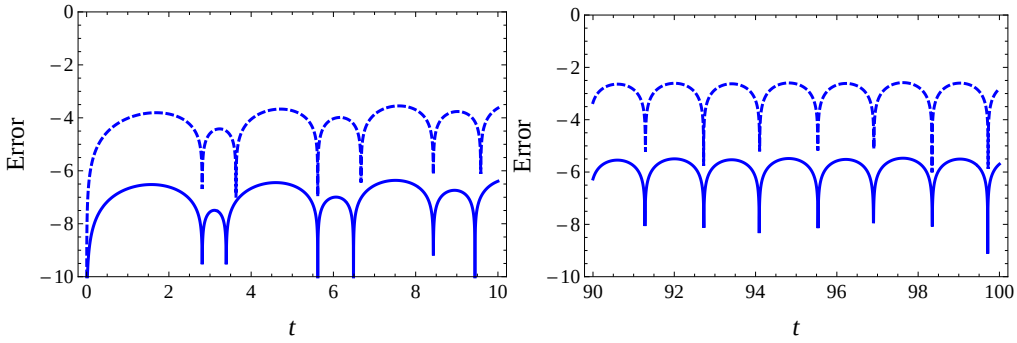
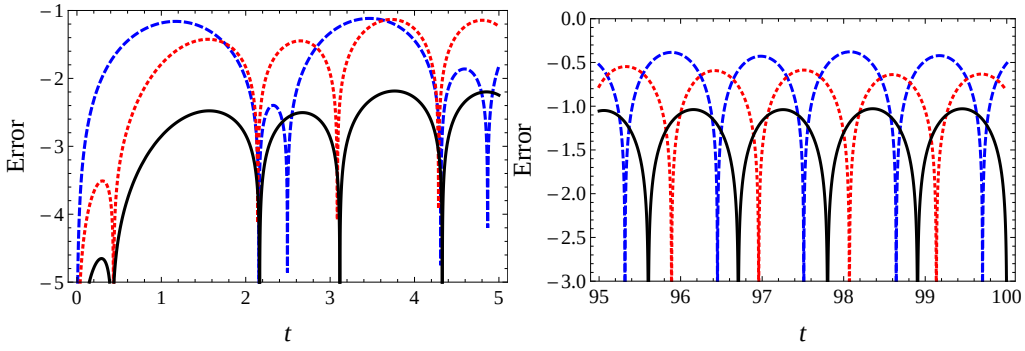


Figure 3: Error in the transition probability (in logarithmic scale) of the approximation of order ε^7 (dashed line), and ε^{15} (solid line) for the periodic example (37)-(38) obtained with algorithm (28)-(29). Two different time intervals are considered for the same integration: $t \in [0, 10]$ (left) and $t \in [90, 100]$ (right).



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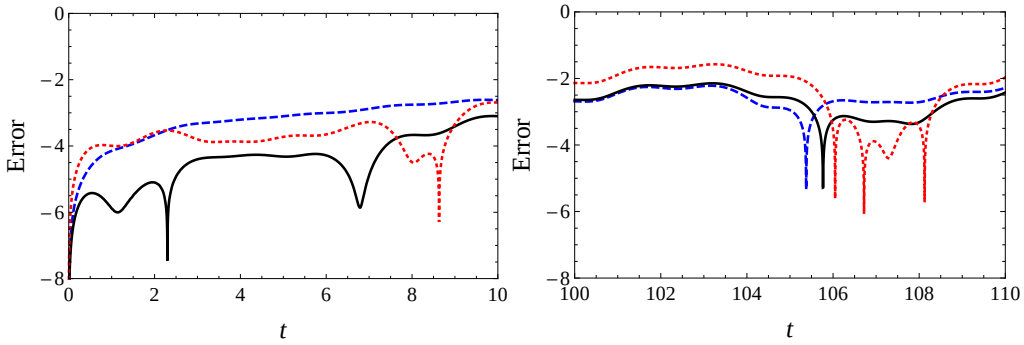

Figure 5: Error in logarithmic scale with three successive transformations up to order ε^7 (solid line), up to order ε^5 (dotted line) and scheme (28)-(29) (dashed line) up to order ε^7 for the Hamiltonian (47). The parameters are $\omega_0 = 2$, $\omega = \sqrt{3}$ and $\varepsilon = 0.6$.