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CONTROL OF CHAOTIC IMPACTS

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We apply controlling chaos techniques to select the desired sequence of impacts in a map that captures universal properties of impact oscillators near grazing. For instance, we can choose the period and then stabilize an unstable periodic orbit with, say, one impact per period involved in the grazing bifurcations that take place in the system.

1. Introduction

In recent years there has been a growing interest in the study of impact oscillators both in the engineering and mathematical literature. This is the term commonly used to represent forced vibrating mechanical systems which also undergo an intermittent or a continuous sequence of contacts with motion limiting constraints. The motion of the components of the system is then a combination of a smooth motion governed by a differential equation interrupted by a series of non-smooth collisions. Even if the smooth motion is linear, the constraints introduce nonlinearity into the overall system. Impact oscillators can be used to model a variety of different systems arising in engineering. Examples include ships moored colliding with fenders, rattling gears and other forced mechanical systems with clearances [Foale & Bishop, 1992].

Mathematically, impact oscillators constitute a subclass of the dynamical systems that do not satisfy the usual smoothness assumptions. These discontinuities are responsible for new forms of behavior [Nordmark, 1991; Budd & Dux, 1994] not found in smooth dynamical systems, particularly in the limit of low velocity or grazing impacts [Foale & Bishop, 1992; Budd & Dux, 1994; Shaw & Holmes, 1983; Shaw, 1985; Nusse & Yorke, 1992; Budd *et al.*, 1995].

In engineering, systems are modeled and investigated to identify, and thus avoid, unacceptable responses. In the case of impacting systems, parametric studies are necessary in order to identify, in particular, regions with high velocity impacts which cause the greatest wear or damage to its components. It is also important to locate regions where chaotic solutions exist so as to avoid the irregular nature of the resulting motion. This could be accomplished, for instance, by the well-known techniques of controlling chaos [Ott *et al.*, 1990]. On the other hand, the same flexibility provided by chaos allows us to select particular trajectories with a desirable sequence of impacts produced in an arbitrary order. This would be advantageous in many technological applications of impact oscillators.

In this paper we apply the method by Ott *et al.* [1990] to control chaotic impacts in the so-called Nordmark map $(x_{n+1}, y_{n+1}) = F_\rho(x_n, y_n)$ [Nordmark, 1991; Chin *et al.*, 1994, 1995], with

$$F_\rho(x, y) = \begin{cases} (\alpha x + y + \rho,\ -\gamma x) & \text{for } x \leq 0 \\ (-\sqrt{x} + y + \rho,\ -\gamma\tau^2 x) & \text{for } x > 0 \end{cases} \tag{1}$$

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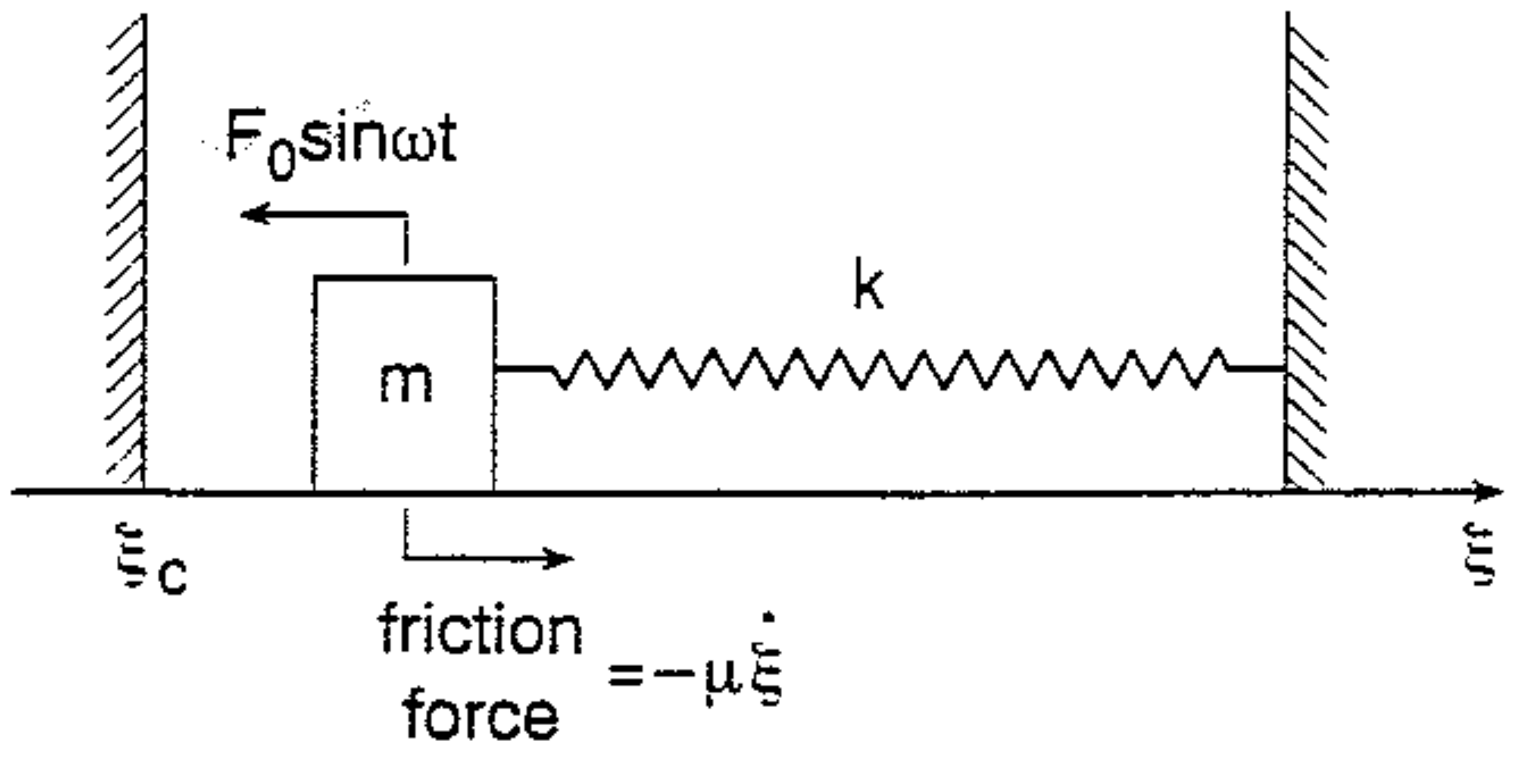


Fig. 1. Physical system modeled by the Nordmark map near grazing.

This is a piecewise continuously differentiable map from $\mathcal{R}^2$ into itself that models the behavior of the sinusoidally forced linear oscillator experiencing impacts at a hard wall as shown in Fig. 1. It is obtained by expanding (to first order) solutions of the system in the neighborhood of a grazing orbit [Nordmark, 1991], i.e. of an orbit that just touches the wall with zero velocity. Thus the map is expected to capture the universal properties of the dynamics in the regime of low velocity impacts. The equivalence with the physical system is established as follows: x_n and y_n are transformed coordinates in the position-velocity space $(\xi, \dot{\xi})$ of the impact oscillator evaluated at times $t_n = 2n\pi/\omega$, where ω is the frequency of the external forcing. The quantity τ^2 is the restitution coefficient of the impacts, whereas ρ is related to F_0, the amplitude of the external force. The parameters α and γ depend on the intrinsic properties of the oscillator in such a way that the limit $\gamma \to 0$ corresponds to a large friction coefficient μ and $\gamma\tau^2 = 1$ gives the opposite limit of zero dissipation. For physical systems (with positive friction) we have [Chin *et al.*, 1994, 1995; Casas *et al.*, 1995]

$$0 < \gamma < 1 , \quad -2\sqrt{\gamma} < \alpha < 1 + \gamma . \qquad (2)$$

The linear part of the map (1) governs the system if there is no impact between time t_n and t_{n+1}. Otherwise, if an impact takes place between t_n and t_{n+1}, the system is described by the second equation. Thus, the effect of impacts in the system is modeled by a square root nonlinearity. The Nordmark map is a stroboscopic map. It captures every asymptotic steady state of the physical system, although there is no direct information in the position of a point with respect to the severity of the impacts.

The map (1) is continuously differentiable except at $x = 0$, where its Jacobian matrix is not defined. In this system the grazing state corresponds to $\rho = 0$, so it describes the dynamics of an orbit in the neighborhood of grazing if $|\rho| \ll 1$, and grazing bifurcations occur as the parameter ρ is increased through $\rho = 0$ with γ and α held fixed. These bifurcations, along with the regions in the (γ, α) parameter space where they take place, have been studied by Chin *et al.* [1994, 1995], and Casas *et al.* [1995]. In particular, for parameter values in the region

$$4\gamma + \frac{1}{4} < \alpha < \frac{3}{2}\gamma + \frac{2}{3} , \qquad (3)$$

it has been shown that there is a grazing bifurcation from a stable period-1 attractor to a reversed infinite period adding cascade, with chaos between successive windows, whereas in the region

$$\frac{3}{2}\gamma + \frac{2}{3} < \alpha < 1 + \gamma \qquad (4)$$

there is a bifurcation from a stable period-1 orbit in $\rho < 0$ to a chaotic attractor as ρ increases through zero. Thus, for $\alpha > 4\gamma + 1/4$, there is an immediate jump to chaos as part of an orbit grazing at the hard wall and, as a result of the bifurcation, the system undergoes a rapid jump to motions with large amplitude [Foale & Bishop, 1992]. The resulting trajectory involves many impact events distributed in a chaotic way.

2. Control of Impacts

Eventually, one may be interested in controlling the irregular behavior that takes place in the Nordmark map. A possible mechanism for doing so (without altering significantly the system) is to apply some controlling chaos technique. In this respect, the approach by Ott *et al.* [1990] has the unique feature that it enables one to select a predetermined time-periodic behavior embedded in a chaotic attractor by making only small time-dependent perturbations to a parameter of the system. The basic idea is as follows [Ott *et al.*, 1990; Romeiras *et al.*, 1992]. First one chooses an unstable periodic orbit embedded in the chaotic attractor according to some specific criterion. Second, one defines a small region around the desired periodic orbit. A trajectory starting with almost any initial condition eventually falls into this small region due to the ergodicity of the chaotic attractor. When this occurs one applies feedback control to force the trajectory to approach the unstable periodic orbit. This method is very flexible because it allows for the stabilization of different periodic orbits for the same set of nominal values of the parameter. For the Nordmark map, we choose ρ to be the accessible control parameter because it characterizes the strength of the driving periodic forcing.

When we apply the algorithm of Ott *et al.* [1990] to the Nordmark map, one can, in principle, stabilize periodic trajectories with an arbitrary number and distribution of impacts per period. This is so even if it is not possible to get analytic expressions for the position of the points along the periodic orbits (for instance, the necessary information needed for applying control can all be extracted purely from observations [Ott *et al.*, 1990; Romeiras *et al.*, 1992]). Here, for simplicity, we consider only maximal periodic orbits [Chin *et al.*, 1994], i.e. periodic trajectories for which there is exactly one impact per period. In this case it is easy to determine analytically the position of the orbit (only one point per period is in the region $x > 0$). For a maximal orbit of period M we can assume, without loss of generality, $x_1 > 0$, so that $x_2, x_3, \ldots, x_M$ are negative and $x_{M+1} = x_1$. The positions of the points along the trajectory can be obtained upon repeated iteration of Eq. (1) and are given by

$$x_{k+1} = \frac{1}{\lambda_1 - \lambda_2}[(\lambda_1^k - \lambda_2^k)(y_1 - \sqrt{x_1}) - \gamma\tau^2(\lambda_1^{k-1} - \lambda_2^{k-1})x_1 + c_0] \quad (5)$$

$$y_{k+1} = \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2}[(\lambda_2^{k-1} - \lambda_1^{k-1})(y_1 - \sqrt{x_1}) - \gamma\tau^2(\lambda_2^{k-2} - \lambda_1^{k-2})x_1 + d_0] \quad (6)$$

with $k = 1, 2, \ldots, M$ and

$$\lambda_1 = \frac{1}{2}(\alpha + \sqrt{\alpha^2 - 4\gamma}), \quad \lambda_2 = \frac{1}{2}(\alpha - \sqrt{\alpha^2 - 4\gamma})$$

$$c_0 = \rho\left[\frac{\lambda_1}{1-\lambda_1}(1 - \lambda_1^k) - \frac{\lambda_2}{1-\lambda_2}(1 - \lambda_2^k)\right]$$
$$d_0 = \rho\left(\frac{1-\lambda_2^k}{1-\lambda_2} - \frac{1-\lambda_1^k}{1-\lambda_1}\right). \quad (7)$$

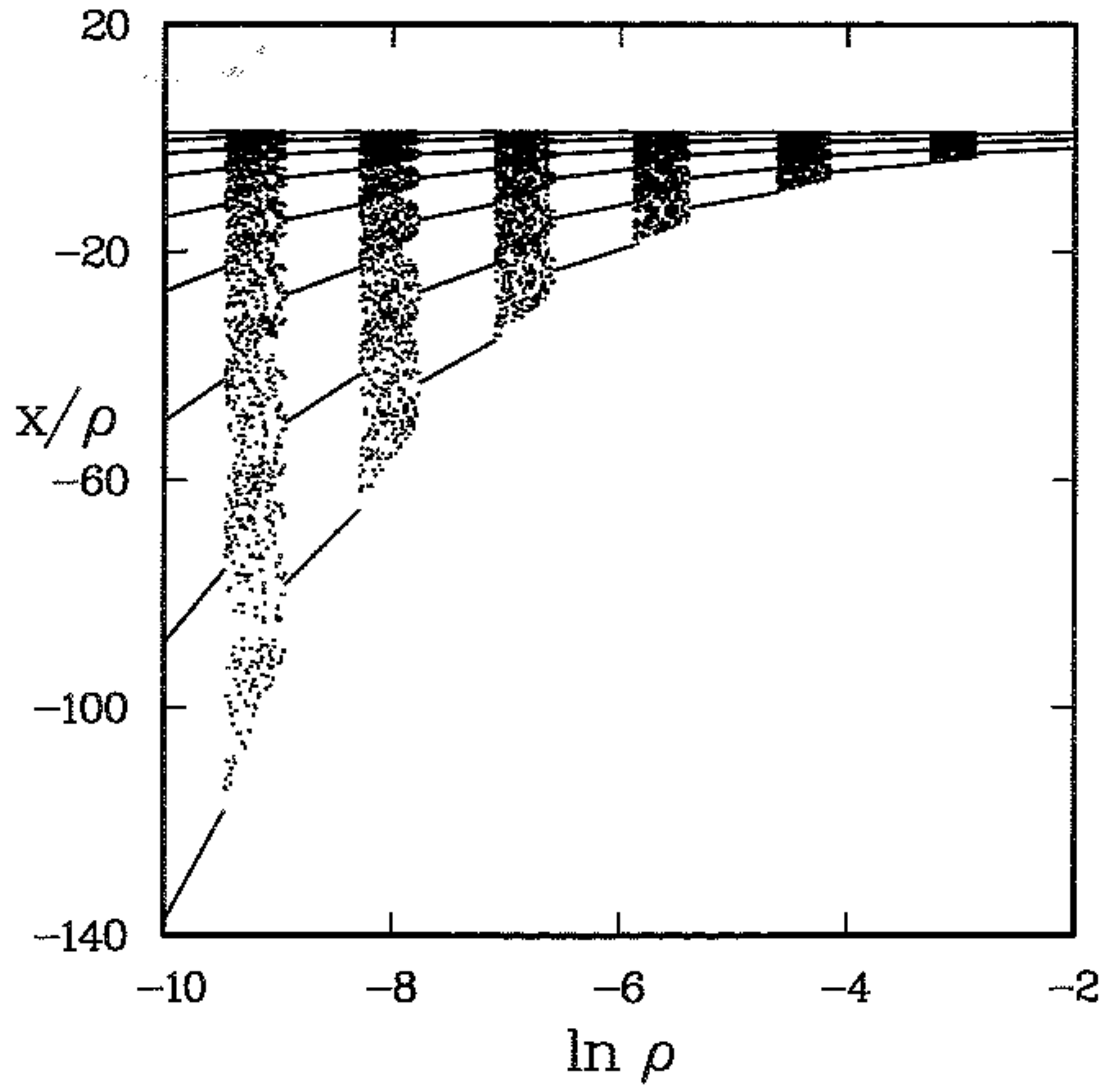
If we make $x_{M+1} = x_1$ and $y_{M+1} = y_1$ in Eqs. (5) and (6) we obtain a quadratic equation for $\sqrt{x_1}$, whose positive solution gives us the coordinate x_1 of the maximal orbit. From Eq. (5) it can be shown that $x_2 < x_3 < \cdots < x_M$. Hence a period-M maximal orbit can exist only if $x_M < 0$, or equivalently, by Eq. (1), if $y_{M+1} = y_1 = -\gamma\tau^2 x_M > 0$ [Chin *et al.*, 1994]. This leads to the existence condition for a period-M maximal orbit

$$y_1 = \sqrt{x_1} + \frac{\lambda_1 - \lambda_2 + \gamma\tau^2(\lambda_1^{k-1} - \lambda_2^{k-1})}{\lambda_1^k - \lambda_2^k}x_1 - \frac{c_0}{\lambda_1^k - \lambda_2^k} > 0. \quad (8)$$

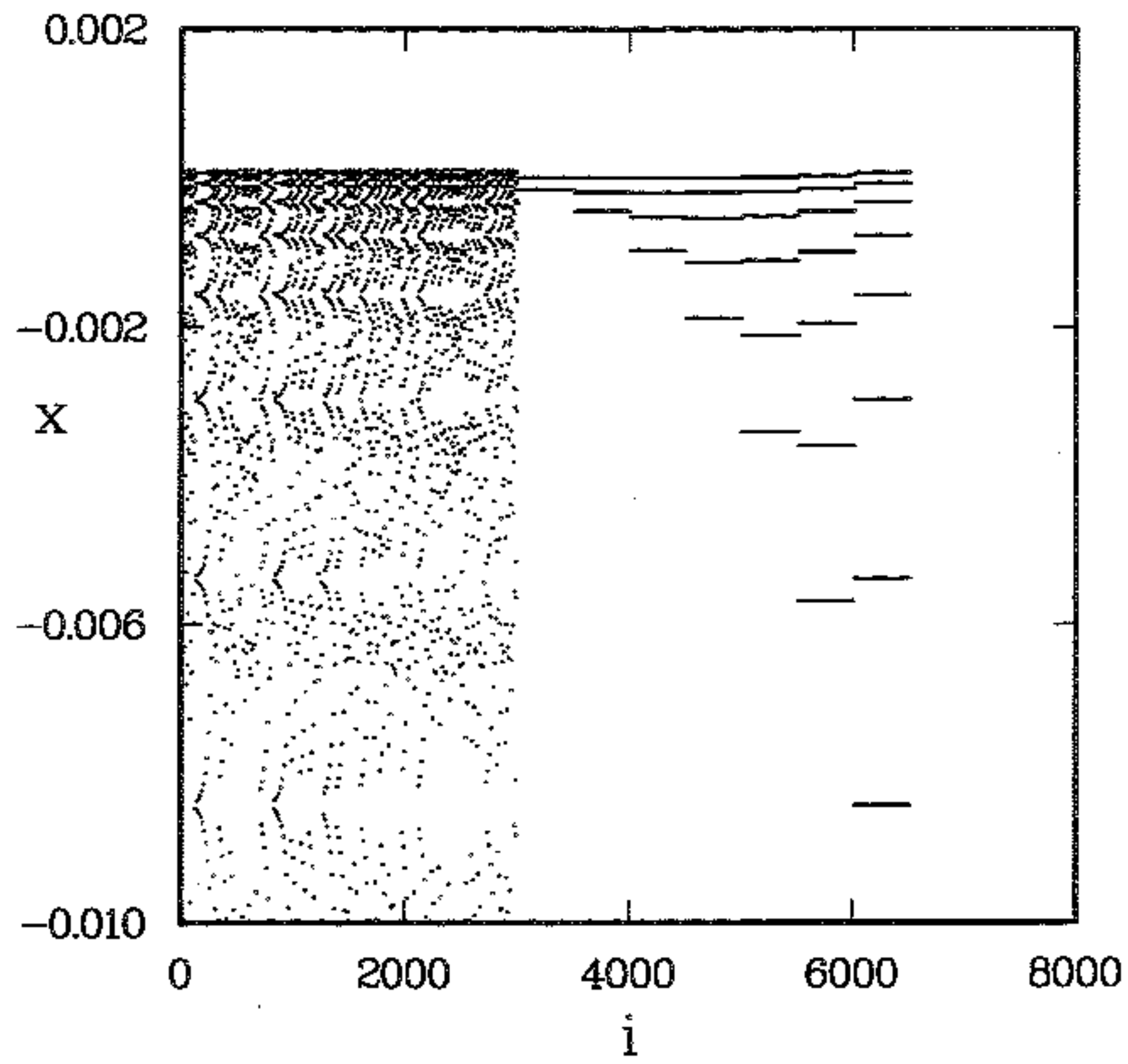
From Eqs. (5) and (6) it is possible to construct the Jacobian matrix and to compute both the eigenvalues of the periodic orbit and the corresponding eigenvectors at each point of the trajectory. Thus, when $\alpha > 4\gamma + 1/4$ we are ready to apply the control algorithm of Ott *et al.* [1990] to stabilize these unstable maximal orbits for small positive values of ρ.

In particular, for systems with parameters in the region $4\gamma + \frac{1}{4} < \alpha < \frac{3}{2}\gamma + \frac{2}{3}$, it was found [Chin *et al.*, 1994, 1995] that as ρ is decreased from positive values, windows of stable maximal periodic orbits are encountered, and each such window is followed by a band of chaos and then by a window of a stable maximal orbit whose period is one higher then the period in the previous window. As ρ decreases, there is an infinite cascade of such windows of decreasing width in ρ and increasing period, accumulating on $\rho = 0^+$. This is illustrated by the bifurcation diagram of Fig. 2(a), obtained for $(\gamma, \alpha) = (0.05, 0.65)$ and $\tau^2 = 1$ for small positive ρ values. Here we can avoid the presence of chaotic impacts for $\rho > 0$ by applying control. As an example we take $\rho = \exp(-9.2)$, on the left band of chaos in Fig. 2(a). Here we have unstable maximal orbits up to period $M = 8$ embedded in the chaotic attractor.

Figure 2(b) illustrates the control of these periodic orbits. The control of the $M = 2$ maximal orbit was turned on after 3000 iterations. We plot the x-coordinate of a trajectory as a function of (discrete) time. The parameter perturbations were programmed to successively control seven different periodic orbits. We switched the control from stabilizing one periodic orbit to another after 500 iterations. The figure shows that the time to achieve control is almost negligible in this case (no apparent transients between switches). Here the maximum allowed parameter perturbation is $\delta = 10^{-4}$. We observe that it is possible to convert a chaotic impacts regime to periodic orbits with only one



(a)

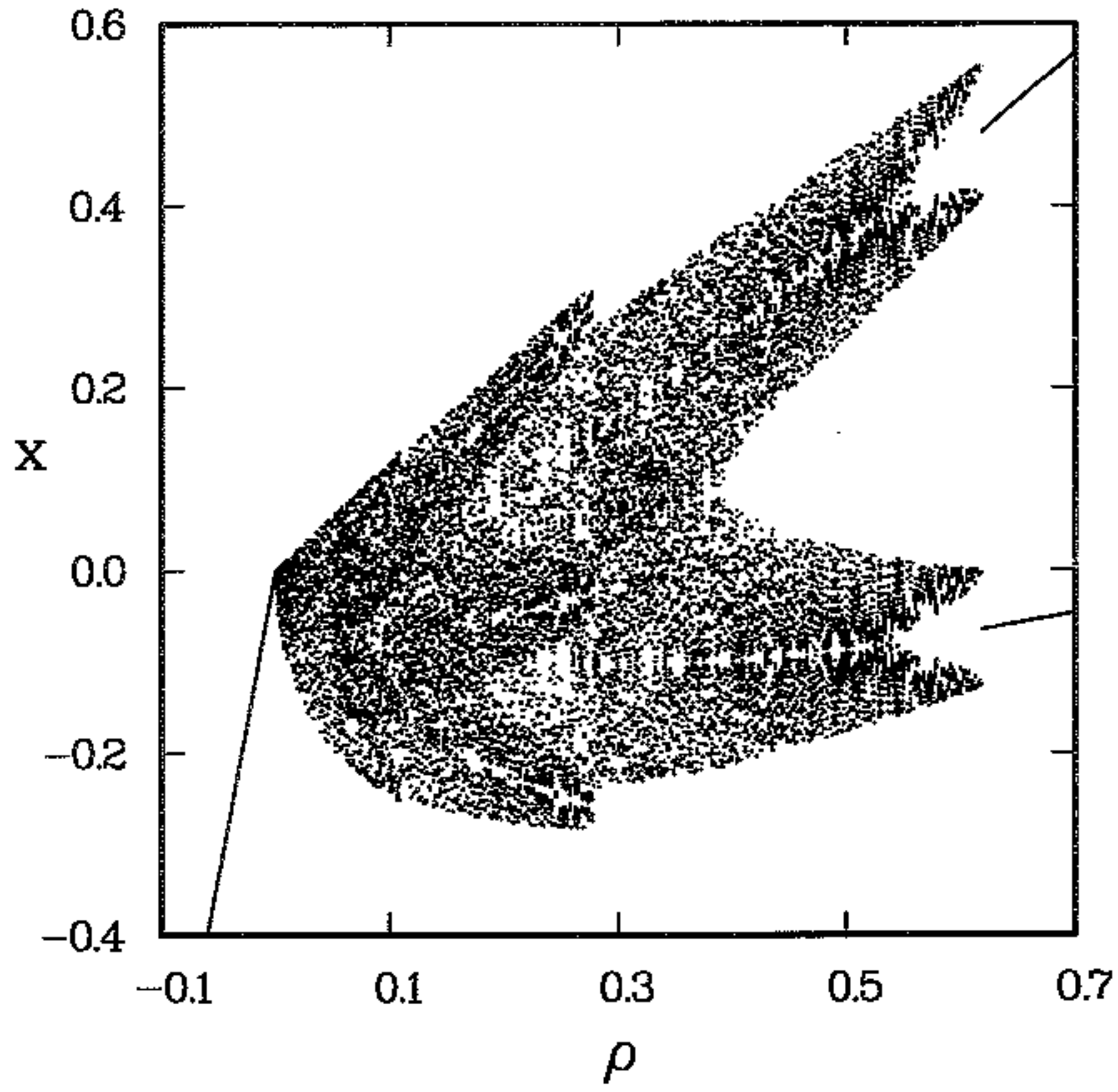


(b)

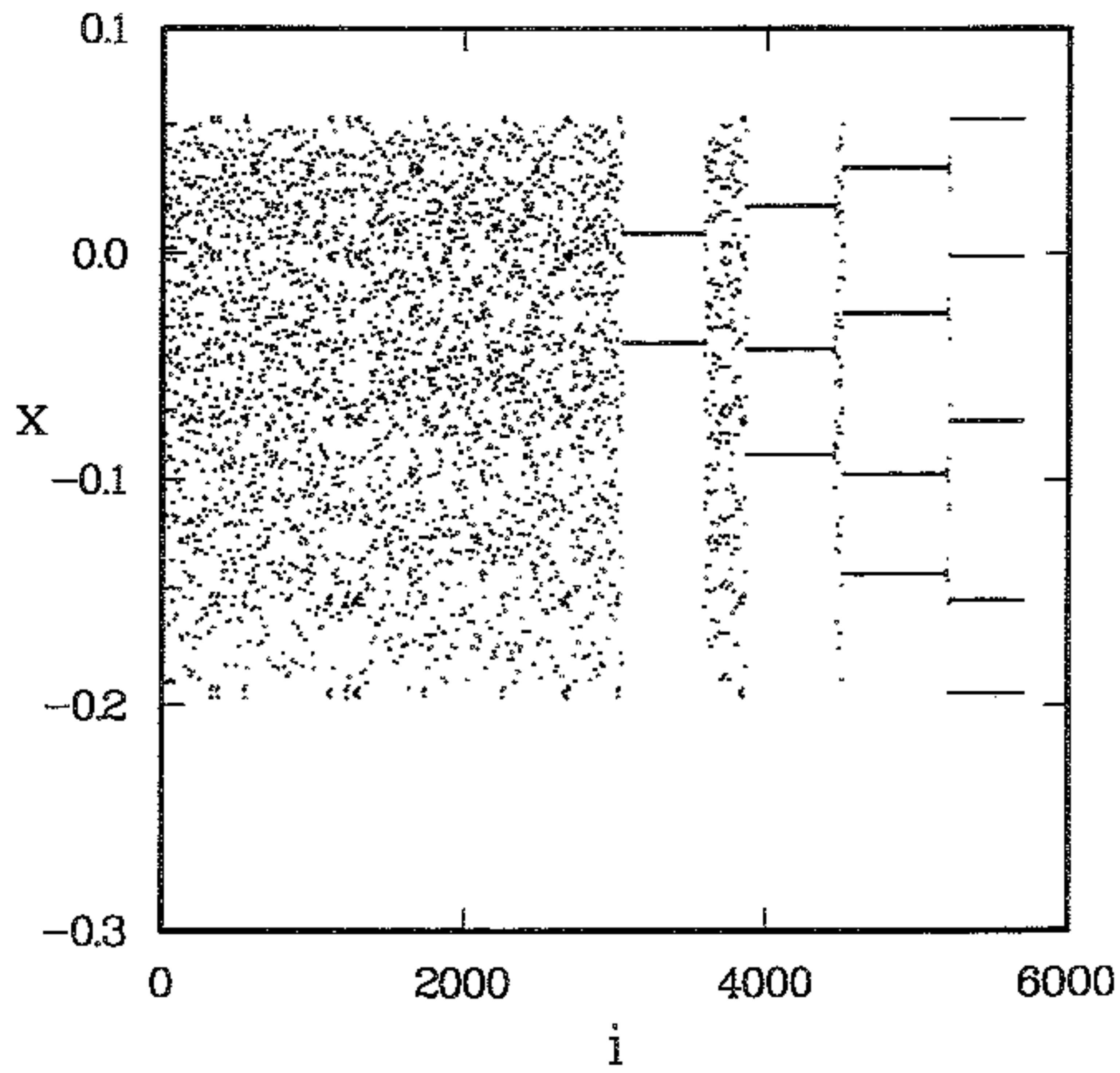
Fig. 2. (a) Bifurcation diagram for $(\gamma, \alpha) = (0.05, 0.65)$ and $\tau^2 = 1$ for small positive ρ values. (b) Successive control of unstable maximal periodic orbits for $\rho = \exp(-9.2)$, starting with period $M = 2$. The maximum parameter perturbation is $\delta = 10^{-4}$.

impact per period by applying small perturbations $|\delta\rho| < 10^{-4}$ to the parameter ρ.

When parameter values in the region $\frac{3}{2}\gamma + \frac{2}{3} < \alpha < 1 + \gamma$ are considered, as ρ increases from zero (corresponding to the occurrence of impacts in Fig. 1) there is an interval of ρ values occupied entirely by a chaotic attractor, and this interval terminates at the appearance of a stable maximal orbit of some period M_0 [Chin *et al.*, 1994]. We have,



(a)



(b)

Fig. 3. (a) Bifurcation diagram for $\gamma = 0.15$, $\alpha = 1$ and $\tau^2 = 1$. (b) Successive control of the unstable maximal periodic orbits embedded in the chaotic attractor for $\rho = 0.05$, starting with period $M = 2$. The maximum parameter perturbation is $\delta = 10^{-3}$.

in particular, unstable maximal orbits embedded in the chaotic attractor with increasing period as ρ approaches zero. An example of a bifurcation diagram for this case is shown in Fig. 3(a), obtained for $(\gamma, \alpha) = (0.15, 1)$ and $\tau^2 = 1$. Here we can control chaos by stabilizing any of the maximal orbits which are present for positive values of ρ. In particular, for $\rho = 0.05$ we have unstable maximal orbits up to period $M = 5$. The control of these periodic orbits is illustrated in Fig. 3(b), where a identical procedure as for Fig. 2(b) has been employed. Now the time to achieve control depends on the particular orbit considered, but the same considerations apply.

3. Conclusions

In summary, we have shown that chaotic dynamics in impact oscillators can be converted, by using only small parameter perturbations, to motion on a desired periodic orbit with a given sequence of impacts. More specifically, it is possible to control chaotic impacts on the hard wall for the system shown in Fig. 1 by adjusting slightly the parameter related to the external force. It is also possible to switch the asymptotic behavior from one periodic orbit to another according to some performance criterion. These results could be of interest in technological applications.

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