CHAOTIC SCATTERING IN THE GAUSSIAN POTENTIAL

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1. INTRODUCTION

It is well known that general classical Hamiltonian dynamical systems have as a rule chaotic behaviour. By such a term one usually understands a sensitive dependence on initial conditions which manifests itself in the topology of phase space. For the most studied case of bounded motions this behaviour is detected, for example, by analysing the Poincaré surfaces of section and by calculating Lyapunov characteristic exponents. The question then naturally arises of what are the effects of this complexity on the unbounded motions, i.e., on scattering phenomena. The signature of chaotic dynamics in these scattering regions of phase space has been the object of several papers appeared mainly in the last decade. Although it has been approached from different points of view it is true that both the number of case studies and the agreement over the quantitative characterization of the phenomenon is much less extensive than the corresponding situation for bounded motion.

Different techniques have been proposed to discuss the manifestations of these chaotic features on scattering phenomena. For example, the concept of Poincaré's surface of section, which rests on the recurrent character of bounded trajectories, has been extended (Jung, 1986) to the unbounded case. But by far the most used indicators of irregular scattering are the time delay and deflection angle as functions of the impact parameter: it is usually accepted that wild fluctuations of these functions betray chaotic behaviour. (Eckhardt, 1988) and (Smilansky, 1991) discuss applications in different physical situations of the idea of irregular scattering.

Most detailed analysis of the structure of the scattering process has been carried out (Bleher et al., 1989;1990) for non-relativistic Hamiltonians in two dimensions. The potentials used present several hills between which a set of unstable periodic orbits exists. A trajectory coming from infinity can be trapped in that region originating a singularity in the time delay or deflection angle as a function of initial conditions. In most cases studied so far the potential is given and the initial conditions are varied to cover different parts of phase space. Frequently the energy is used as the param-

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eter which sets the transition from regular to irregular behaviour. The set of initial conditions for which those singularities occur is shown to be fractal in nature.

In this contribution we present preliminary results for a family of simpler potentials which allows one to analyse, for a given set of initial conditions, the transition from a completely integrable situation to the general one. The potential has been used also to study relativistic effects and non conventional momentum dependences in chaotic Hamiltonian systems (Casas, 1989; 1992).

2. THE GAUSSIAN POTENTIAL

We are interested in the non-relativistic motion in two dimensions of a particle of unit mass under the action of the potential

$$V(x,y) = s\exp(-x^2 - qy^2), \tag{1}$$

where s distinguishes between attractive ($s = -1$) and repulsive ($s = 1$) cases. By variation of the q parameter this potential goes from the central case $q = 1$, which is integrable, to the non-central case ($q > 1$). This is a much simpler case than most of the potentials considered in the literature. Nevertheless, as we will show, it originates a very complex dynamics. The Hamiltonian of our problem will be

$$H = \frac{1}{2}(p_x^2 + p_y^2) + s\exp(-x^2 - qy^2), \tag{2}$$

where all the variables are dimensionless. For a wide spectrum of q values and initial conditions this system has shown to have all the characteristics of non-integrability for bounded motions (Casas, 1989), and here we will be interested in the study of the unbounded region of phase space. For that purpose we will have to integrate numerically the equations of motion. This will be done by using a Taylor series up to the seventh order, a scheme which has been proved to be very accurate and stable for this problem.

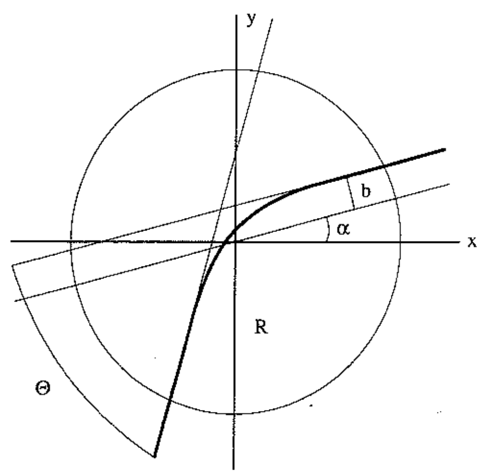

Figure 1. Schematic of a scattering trajectory.

A typical scattering trajectory is schematized in Fig. 1. A particle coming from infinity with energy $E > 0$, momentum $\vec{p}$ and angular momentum $\vec{l}$ enters a circle of radius R with incidence angle α. We take R large enough to ensure that outside that circle the particle is practically free. In our calculations we have taken $R^2 = 40$ and, due to symmetry reasons, we limit α to the range $[0, \pi/2]$. The impact parameter is defined by $b = |\vec{l}|/|\vec{p}|$. Each set of values of E, R, α and b determines uniquely a trajectory.

The magnitudes we are interested in are the deflection angle Θ and the time delay T, defined as the difference between the time it takes to the particle to traverse the region $x^2 + y^2 < R^2$ when the potential is on and the time taken when the particle is free.

The computational procedure to calculate $\Theta(b)$ and $T(b)$ consists in, keeping fixed the values of the energy E and the incidence angle α, dividing the interval $[b_{min} = -R, b_{max} = +R]$ in a number of points. For each b in this interval the equations of motion are integrated till the particle exits the circle of radius R, the scattering angle Θ being determined by the final momentum.

3. DISCUSSION OF THE RESULTS

Repulsive case

This corresponds to $s = 1$ and the form of the potential obviously does not allow the existence of periodic orbits, and so no irregular scattering is possible. Nevertheless, there is a peculiar phenomenon worth to be mentioned. In the central case ($q = 1$) a particle with $b = 0$ and energy $E = 1$, which is the top of the potential hill, asymptotically reaches the origin giving a divergence in the time delay function.

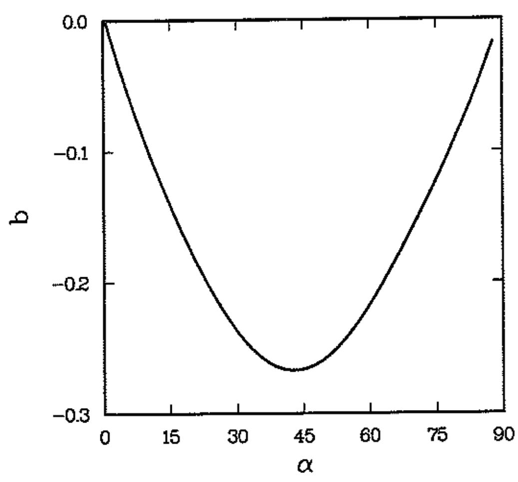

Figure 2. Distribution of points (b, α) for which a singularity appears in the functions $T(b)$ and $\Theta(b)$ for $q = 3$, $E = 1$ and $s = 1$.

For elliptical hill tops it has been argued (Bleher et al., 1989;1990) that, independently of the incidence angle α, the scattering angle Θ as a function of b presents a singularity in $b = 0$ for $E = 1$, and this behaviour is claimed to be a universal one for these potential shapes at the hill top energy. On the contrary, our results point in a different direction. In fact, we obtain that for $E = 1$, $\Theta(b)$ presents singularities at values of b which depend on the incidence angle α. In Fig. 2 we show the whole set of singularities b as a function of α for $E = 1$, $q = 3$. In particular, only for $\alpha = 0$ is $b = 0$ a singularity.

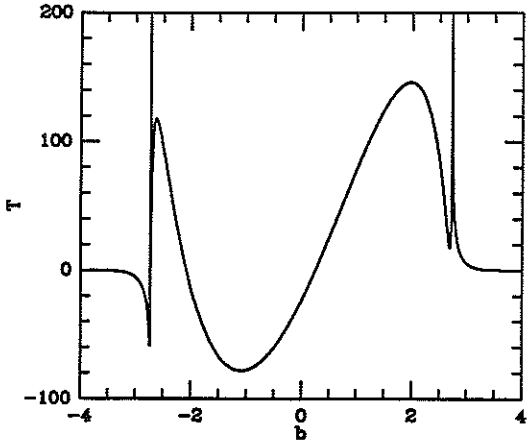

Figure 3: Time delay as a function of the impact parameter for $q = 1$, $E = 0.01$ and attractive potential ($s = -1$).

Attractive case

As an illustrative situation of this case we report the results obtained with $q = 1$ and $q = 3$ for the time delay and the scattering angle as functions of b for $E = 0.01$ and $\alpha = \pi/3$. Obviously, in the central potential there is no dependence on α.

Figure 3 presents the time delay $T(b)$ for $q = 1$. We can observe two discontinuities in that figure, which correspond to the well known phenomenon of orbiting (Newton, 1982). They appear symmetrically with respect to the origin, in agreement with the general theory. Furthermore, these particular values of b have been checked by an independent numerical procedure from the very definition of orbiting.

Figure 4 shows the corresponding results for $q = 3$. We observe a much intricate behaviour of the function $T(b)$ with a huge number of singularities distributed along an interval on the b-axis. To illustrate the possible fractal character of this set, a blowup of it is shown in Figure 5.

Instead of keeping fixed the energy and incidence angle and vary the impact parameter, we can fix b and study the dependence of T with E. Now the results show a 'resonance' structure similar to the quantum mechanical scattering problem. A typical illustration of this behaviour is given in Fig. 6.

Although the previous comments refer to the plots T vs. b, the same structure has been obtained for Θ vs. b.

4. CONCLUSIONS

A very simple potential with elliptic equipotential contours of varying ellipticity has been proposed to study the onset of chaotic scattering. Both repulsive and attractive cases have been considered. In the former case we have shown that for the hill top

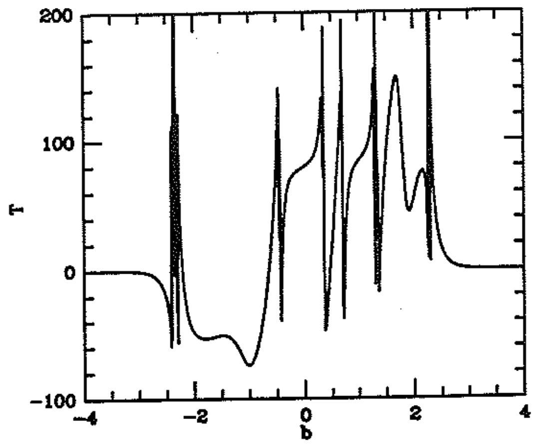

Figure 4. Time delay as a function of the impact parameter for the non-central case $q = 3$, $E = 0.01$, $\alpha = \pi/3$ and $s = -1$.

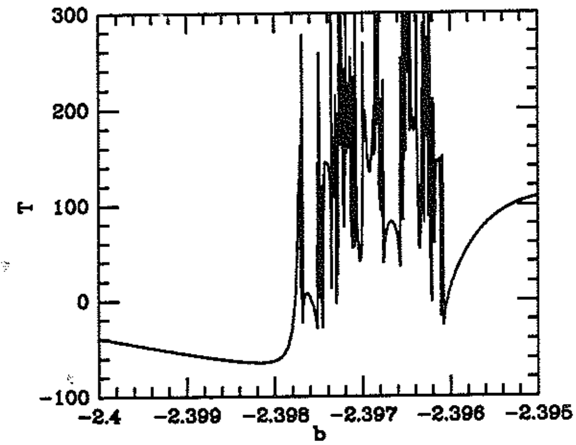

Figure 5. Blowup of the function $T(b)$ for $q = 3$, $E = 0.01$, $\alpha = \pi/3$ and $s = -1$.

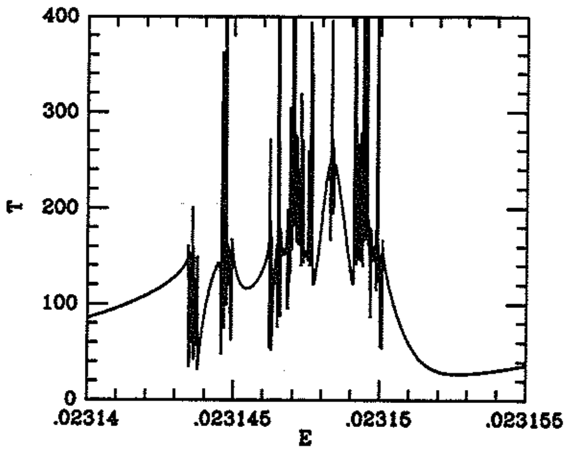

Figure 6. Time delay as a function of the energy for the non-central case $q = 3$, $\alpha = \pi/3$, $s = -1$ and impact parameter $b = 2$.

energy the scattering angle presents singularities which are movable with the incidence angle, at variance with what has been stated elsewhere. For the attractive potential we have observed the typical behaviour in the singularity structure of the time delay and deflection angle functions. Further work in this case to study the dependence of the fractal dimension of that set of singularities on the non-integrability parameter q is in progress.

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