# Adapting Splitting Methods to Stellar Dynamics

## A Wishlist

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# Gravitational *N*-body Problem

Solving the equations of motion, for a large number of particles, moving under the influence of the gravity of each other:

$$\mathbf{F}_{ij} = G\frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}(\mathbf{r}_i - \mathbf{r}_j)$$

Dynamical evolution of planetary systems, star clusters, galaxies, galaxy clusters...

# Gravitational *N*-body Problem

Hamiltonian:

$$\mathcal{H} = \sum_{i}^{N} \frac{\mathbf{p}_i^2}{2m_i} - G \sum_{i}^{N} \sum_{j>i}^{N} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|}$$

# Gravitational *N*-body Problem

Hamiltonian:

$$\mathcal{H} = \underbrace{\sum_{i}^{N} \frac{\mathbf{p}_i^2}{2m_i}}_{\mathcal{T}(p)} \underbrace{- G \sum_{i}^{N} \sum_{j>i}^{N} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|}}_{\mathcal{U}(q)}$$

Applying splitting methods (or RKN methods) is straightforward

# Star Clusters

Distance between the stars can vary wildly:

- Close encounters
- Density diffences

➨ Different rates of evolution

➨ Individual, adaptive timesteps!

# Individual Adaptive Timesteps

Solve:

$$\frac{d^2\mathbf{r}_i}{dt^2} = -G\sum_{j\neq i}\frac{m_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}(\mathbf{r}_i - \mathbf{r}_j)$$

for each star seperately.

We need a way to estimate positions of other stars cheaply.

# Individual Adaptive Timesteps

Predictor-corrector methods seem like a natural choice.

Hermite interpolation is the most commonly used scheme, since it provides accurate interpolating polynomials.

$$\mathbf{a}_0\,,\ \mathbf{j}_0 = \left.\frac{d\mathbf{a}}{dt}\right|_{t=0}\,,\ \mathbf{s}_0 = \left.\frac{d^2\mathbf{a}}{dt^2}\right|_{t=0}\,,\ \ldots$$

$$\mathbf{r}(t) = \mathbf{r}_0 + \delta t\,\mathbf{v}_0 + \frac{\delta t^2}{2}\mathbf{a}_0 + \frac{\delta t^3}{6}\mathbf{j}_0 + \ldots$$

# Individual Adaptive Timesteps

Predictor-corrector methods seem like a natural choice.

Hermite interpolation is the most commonly used scheme, since it provides accurate interpolating polynomials.

This scheme also provides a way to calculate an appropiate timestep.

# Wishlist (I)

- Estimating positions of the stars at arbitrary time.
  (dense output? history extrapolating?)
- Estimating errors ➨ determining timestep
  (embedded methods? complex timesteps with purely imaginary leading error?)

# Motivation

- Efficiency:

$$[\mathcal{V}, [\mathcal{V}, [\mathcal{V}, \mathcal{T}]]] = 0$$

Could exploiting this structure increase efficiency?

# Motivation

- Efficiency
- High order:

  “Exact” integrations are being studied, to understand chaos, entropy etc.

  $N \sim 100$

  ➨ High orders methods seem necessary.

# Motivation

- Efficiency
- High order
- Hybrid force calculations:

  We study very large dynamical scales.

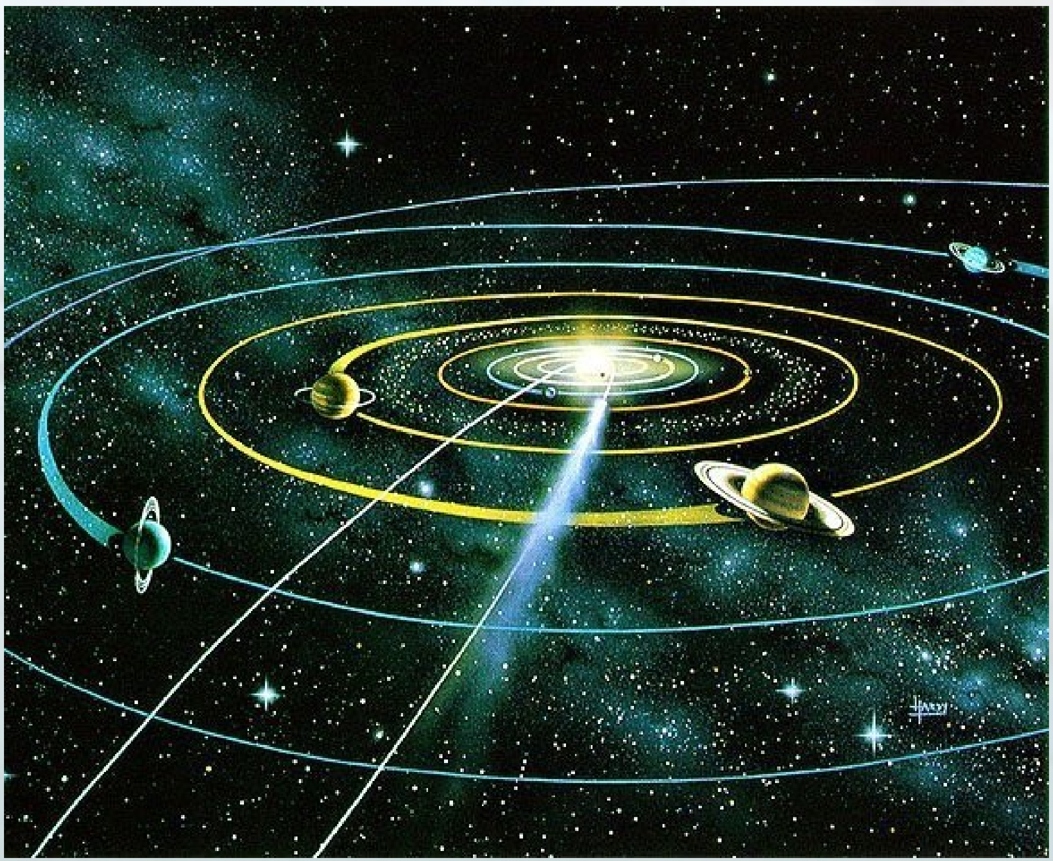
  It is clear that brute force cannot work.

  We have to mix rapid/slow forces and accurate/approximate force calculations.

# $N$-body Problem Revisited

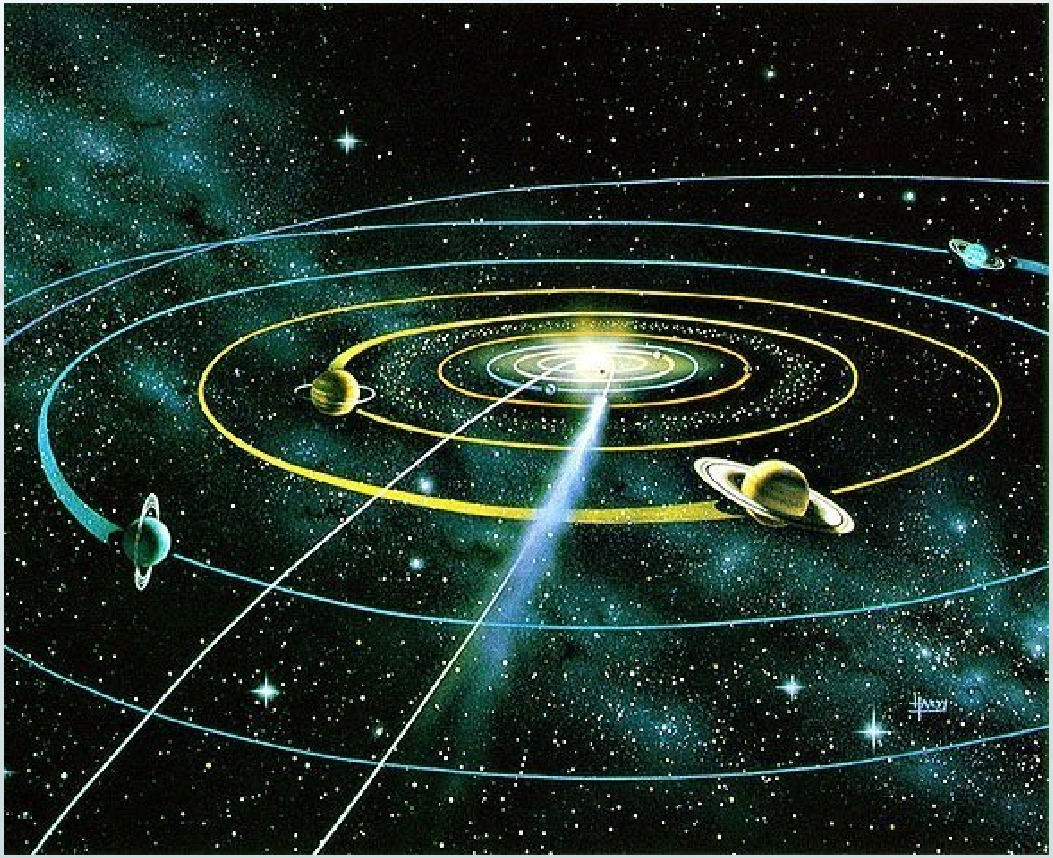
# *N*-body Problem Revisited

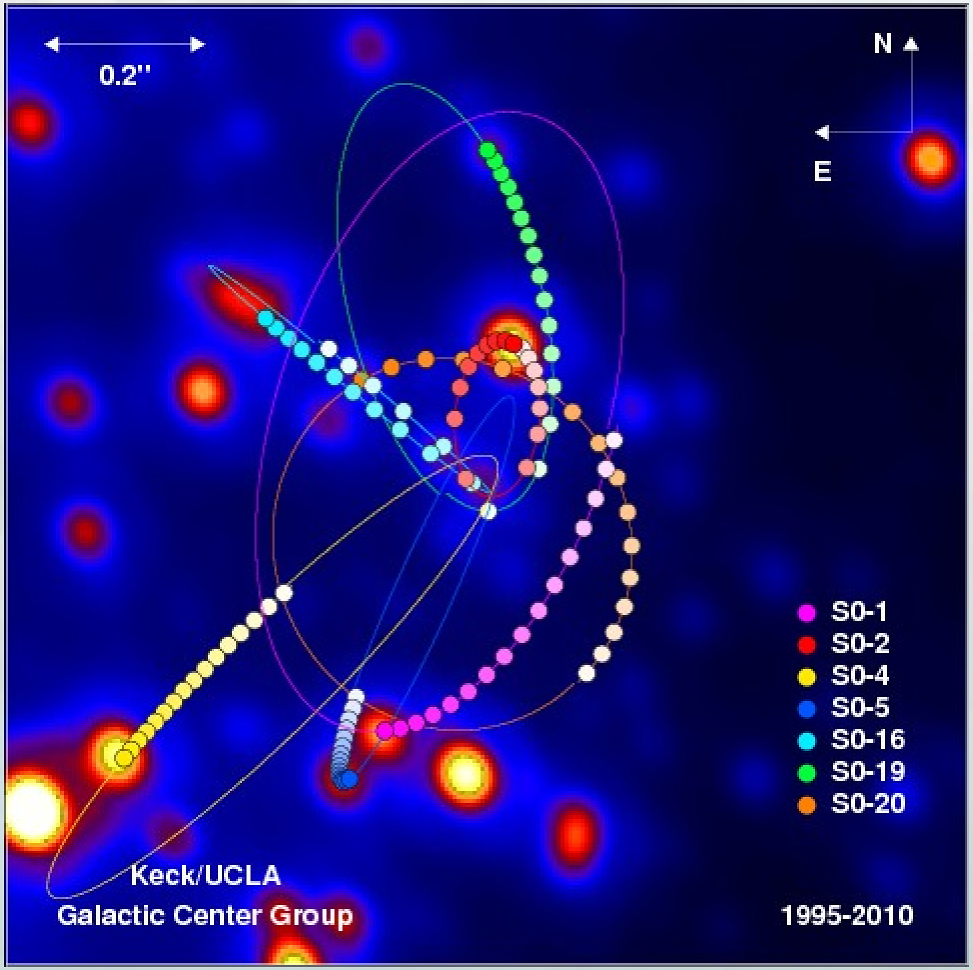
Near-Keplerian systems:



# *N*-body Problem Revisited

Near-Keplerian systems:





# Near-Keplerian Systems

Rewriting the Hamiltonian for near-Keplerian systems:

$$\mathcal{H} = \sum_{i}^{N} \frac{\mathbf{p}_i^2}{2m_i} - G\sum_{i}^{N} \frac{Mm_i}{|\mathbf{q}_i|} - G\sum_{i}^{N}\sum_{j>i}^{N} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|}$$

$$+\frac{1}{2M}\left(\sum_{i}^{N} \mathbf{p}_i\right)^2$$

# Near-Keplerian Systems

Rewriting the Hamiltonian for near-Keplerian systems:

$$\mathcal{H} = \underbrace{\sum_{i}^{N} \frac{\mathbf{p}_i^2}{2m_i} - G\sum_{i}^{N} \frac{Mm_i}{|\mathbf{q}_i|}}_{\mathcal{K}(q,p)} - \underbrace{G\sum_{i}^{N}\sum_{j>i}^{N} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|}}_{\mathcal{I}(q)}$$

$$\underbrace{+\frac{1}{2M}\left(\sum_{i}^{N} \mathbf{p}_i\right)^2}_{\mathcal{S}(p)}$$

# Near-Keplerian Systems

If $M \gg m_i$ , $\mathcal{S}(p)$ can be neglected.

In this case, optimized RKN methods work beautifully.

$$[\mathcal{I}, [\mathcal{I}, [\mathcal{I}, \mathcal{K}]]] = 0$$

# Near-Keplerian Systems

General case, should also allow simplifications since

$$[\mathcal{I},[\mathcal{I},[\mathcal{I},\mathcal{K}]]]=0$$

$$[\mathcal{I},[\mathcal{I},[\mathcal{I},\mathcal{S}]]]=0$$

$$[\mathcal{K},[\mathcal{K},[\mathcal{K},\mathcal{S}]]]=0$$

# Near-Keplerian Systems

Surprisingly, the only splitting used in the literature is a generalization of leapfrog:

$$\exp(\frac{1}{2}\mathcal{S})\exp(\frac{1}{2}\mathcal{I})\exp(\mathcal{K})\exp(\frac{1}{2}\mathcal{I})\exp(\frac{1}{2}\mathcal{S})$$

# Near-Keplerian Systems

Surprisingly, the only splitting used in the literature is a generalization of leapfrog:

$$\exp(\frac{1}{2}\mathcal{S})\,\exp(\frac{1}{2}\mathcal{I})\,\exp(\mathcal{K})\,\exp(\frac{1}{2}\mathcal{I})\,\exp(\frac{1}{2}\mathcal{S})$$

This works fine when $\mathcal{K} \gg \mathcal{I}$ and $\mathcal{K} \gg \mathcal{S}$ but this is not always the case.

Also, (I suspect) it is not very efficient.

# Wishlist (II)

- Higher order optimized methods for 3-way splitting. These can exploit:
  - computing S(p) is cheap
  - I(q) is small
  - the vanishing commutators
- Methods that specifically supress errors for large S(p)

# Motivation

- Centres of galaxies (as gravitational wave sources)
- Solar system (Oort cloud and early evolution.
- Exoplanets,

These are all exploding fields.

Improvements on the integrators for these would be quite welcome, if they can be implemented easily.

Thank you!